Excerpts from
'On the Conceptions that Underlie Space'

(1790)

Abraham Gotthelf Kästner

ABRAHAM GOTTHELF KÄSTNER was one of the most important figures in the history of science. Born in 1719 in Leipzig, Kästner spent his entire life as a passionate defender of Kepler and Leibniz. In 1756, he was appointed professor of mathematics at Göttingen University, where his students included Carl F. Gauss. In 1766, he hosted Dr. Benjamin Franklin's visit to the University, as part of his continuing support for the efforts of Leibniz's networks to create a republic in North America.

Kästner's contributions to science are numerous, but perhaps his most important was his seminal work on developing what his student Gauss would later call "anti-Euclidean" geometry. As the following excerpts demonstrate, Kästner, like Leibniz and Kepler, rejected Aristotle's insistence that all knowledge must rest on a set of a priori axioms, postulates, and definitions, as is the case in Euclidean geometry. As Kästner indicates, all Euclidean geometry depends on the truthfulness of the so-called "parallel postulate," which must be accepted without proof. Gauss, and his student Bernhard Riemann, adopted Kästner's view, demanding that all such axiomatic systems be discarded, and that physical science be based on demonstrable universal physical principles alone.

25. [The article in] *Phil. Mag.* II, Vol. 1, p. 88, gives the arguments, why the well-known eleventh axiom* of Euclid does not possess the evidence, that an axiom should have: because it contains the concept of an infinity: a space that, in regard to its bounding, would be indefinitely extended, and which the senses and imagination could not represent.

26. Speaking precisely, it does not seem to me that this concept is contained in Euclid's axiom. . . . [When Euclid speaks of] extension of lines into the infinite, this means nothing other than: extending *as far as necessary.* Euclid says only, that [if two lines intersect a third one in such a way, that the inner angles formed add up to less than two right angles, then] one can extend the lines so far, that they intersect. The visual concept of a such pair of lines is thus as follows: When the sum of the two angles comes closer and closer to two right angles, then the lines will stay apart for a longer distance, before they intersect; and the axiom only says: The lines can always be extended far enough [i.e., to the point where they intersect–JT].

27. In the magazine mentioned above, we read, "they intersect, when they are extended to *infinity.*"

This is a correct translation of the [Greek] words, but the Greeks did not attach the same thought to the word, which we call *the infinite,* as what we think—or believe we think—in connection with the German expression.

For, if it were *necessary* to extend the two lines to *infinity* (in the newer sense of that word), in order for them to intersect, then they *do not intersect at all,* but are parallel. . . .

28. One says: Parallels *intersect* at *infinity;* in what sense this is said, I have show in my *Foundations of Geom-*

* I.e., the "parallel postulate." Modern editions number this as the Fifth postulate.–Ed.

etry 12, p. 12. If this manner of speaking is to be defended, then it must certainly signify as much, as to say: the lines *never intersect.*

29. If the two lines (as in 26) intersect each other, then they form together with the third line, where the two angles mentioned above are formed, a *triangle,* and thereby a figure which constitutes a bounded space. In Euclid's expression, "they intersect, when they are continued to infinity," the word *infinity* signifies nothing other, than what we explained in point 26 above.

30. The reason why one does not find the same degree of evidence in this axiom, as in the others, does not lie in the concept of *infinite space* in the newer meaning of the word, but rather: that we have merely a *clear* concept of straight line, and not a *definite* concept, as I pointed out in my article: what it means in Euclid's geometry to be possible.

If two straight lines, in the same plane, are perpendicular to a third line, then they never intersect. This conclusion flows from the *clear* concept of straight line: for, on one side of the third line everything is identical to the other side, and so the two lines would have to intersect on the other side also, if they intersect on this side. But they cannot intersect twice . . . .

31. However, when only one of the two lines is perpendicular to the third, and the other does *not* form a right angle, then do they intersect? And on which side of the third line?

This is a part of the non-evident eleventh axiom; and if one could establish the correctness of this part, then the whole would follow.

32. The lack of evidence does not flow from the fact, that we have no visual concept of an infinite space. . . .

33. The question is this: Let one extend the oblique straight line (31) as far as one wants: do we ever get to the perpendicular line?

The difficulty to give an affirmative answer to this question, comes from the following:

One could imagine, instead of the oblique *straight* line, a *curved* line, whose asymptote would be the perpendicular straight line—for example, a branch of a *hyperbola.*

With such a curved line, one could go out as far as one wanted, without ever touching the perpendicular line.

Why should something *necessarily* occur with an oblique *straight line,* which does not *have* to occur, when one replaces it with a curved line?

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34. Thus, the difficulty concerns the distinction between *curved* and *straight* lines. A curved line means, a line in which no part is straight. This concept of a curved line is *distinct,* because the concept of straight line is *clear;* but it is also *incomplete,* because the concept of straight line is *merely* clear.

35. So we see, that the difficulty of the axiom does not concern *infinite space,* but rather the indistinctness and incompleteness of the concepts.

36. A curved line never intersects its asymptote; the strange thing about this is, that it always approaches toward the asymptote. A bit of thought shows: never intersecting, but always approaching, could only occur at the same time, if the amount of approach becomes smaller and smaller. So, the oblique line (31) surely must approach [the other line] at a constant rate, although not an increasing one. Since this proposition is sufficient, one need only prove it, in order to establish that an intersection occurs. This has also been attempted, but without success, because the required result cannot be derived from concepts that are no more than clear.

37. Besides, in geometry, "asymptote" means nothing more than a straight line that comes ever closer to a curved line, without ever touching it. The more modern authors say: the curved line is intersected by the asymptote at infinity; and even regard the pair of such touching-points, located far across each other on the infinite line, as constituting a single point; and other such paradoxes, which all arise from a poetic interpretation of the simple, clear prose of the ancients. Even the concept of asymptotes requires nothing more, than a space, in which extension can be continued further and further.

38. In Joseph Raphson, *Analysis aequationum universalis,* 2nd edition (London: 1694), one can find his article on "Spatio reali, seu ente infinito, conamen mathematico metaphysicum." In Chapter 3, he calls the infinity of the mathematicians, as this occurs with series, asymptotes, etc., the *potential infinite.* . . .

Raphson says exactly what I have said. Since I have restricted myself to space, as the geometers conceive it, the rest of his propositions don't belong to my present intentions; besides which, I still think the way *Leibniz* and *Wolf* taught me to think.

—translated from the German by Jonathan Tennenbaum