On Principles and Powers

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Rembrandt van Rijn, "Aristotle Contemplating a Bust of Homer," 1653.

Rembrandt van Rijn's masterpiece, *Aristotle Contemplating a Bust of Homer,* conveys a principle that leads directly into the deeper implications of Gauss's and Riemann's complex domain. In the painting, the eyes of both figures are fixed directly before them; yet, Aristotle's gaze is insufficient to guide him. To find his way, he reaches forward to touch the likeness of the poet, who, although blind in life, leads the blocked philosopher in a direction he would otherwise be incapable of finding [SEE illustration, and inside front cover, this issue].

Like the navigators of ancient maritime civilizations, Rembrandt's Homer knows that "straight ahead" is not necessarily where your eyes point. When following a course across some wide expanse, these discoverers would mark their passage by noting the motions of celestial bodies, the which were charted as changes of position on the inside of the sphere whose center was the eye of the observer. When the observer's position changed, so did the position of everything on the sphere; but the manifold of vision remained a sphere, and the eye of the observer remained at its center. A stationary observer would note certain changes in the position of celestial bodies over the course of a night, and from night to night. An observer moving on the Earth, noted these changes, plus the changes in these changes resulting from his own motion. These changes, and changes of changes, formed a map in the mind of the explorer—not a static map, but a map of the principles that caused the map to change. It is the map of principles in which all explorers, from those days to this, place their trust.

While a map, such as one of positions of celestial bodies on the inside of a sphere, can be represented directly to our senses, a map of principles can only be represented by the methods exemplified in Rembrandt's painting. Principles do not appear as objects in the picture, but as ironies that evoke the formation of their corresponding *ideas* in the imagination of the viewer. The scientist in pursuit of unknown principles, must master the art of recognizing the ironies that appear, not only from known principles, but from those yet to be discovered; these latter emerging as paradoxes. In the case of physical principles investigated by mathematical images, these paradoxes present themselves as anomalies, as, for example, the emergence of $\sqrt{-1}$, within the domain of algebraic equations. The poetic scientist takes the existence of such anomalies as evidence of a principle yet to be discovered, and re-thinks how his map must change to include this new principle. C.F. Gauss measured this type of transformation as a change in *curvature.* This work was extended by Bernhard Riemann through his theory of complex functions, most notably in his major works on the hypergeometric and Abelian functions.

What has failed Rembrandt's Aristotle is not his eyes, but his map: a map which has been changed by a principle which, "on principle," Aristotle insists does not exist and could not be known if it did. Disoriented, he is left to grope in the only direction he knows—straight ahead. Fortunately for him, straight ahead stands the lifeless image of Homer, possessed with the power to light his way.

Curvature and Power

This method of discovery is already evident in the work of the Greek geometer Archytas, who taught that the physics of the universe could be discovered by investigating the paradoxes that arise in arithmetic, geometry, spheric (astronomy), and music. His collaborator Plato prescribed mastery of these four branches of one science, as essential to the development of political leadership.

The solution Archytas provided to the problem of doubling the cube exemplifies the principle. Doubling the line, square, and cube, presents us with the existence of magnitudes of successively higher powers, each of which is associated with a distinct principle.*

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* For a full discussion, see Bruce Director, "The Fundamental Theorem of Algebra: Bringing the Invisible to the Surface," *Fidelio,* Summer/Fall 2002, (Vol. XI, No. 3-4).

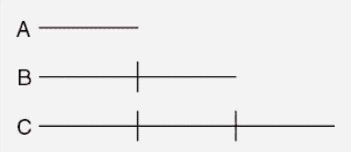

FIGURE 1. *Lines A, B, and C are in arithmetic proportion.*

The Pythagoreans called the power that doubles the line, "arithmetic," and the power that doubles the square, "geometric," which they associated with musical intervals as well as mathematical ones. In their most general form, the arithmetic is associated with the division of a line, whereas the geometric is associated with the division of a circle [SEE Figures 1 and 2). From Gauss's standpoint, the change in power from the arithmetic to geometric, is associated with a change in curvature from rectilinear to circular.

As Archytas's predecessor Hippocrates of Chios knew, to double the cube requires placing two geometric means between two extremes. At first approximation, this can be accomplished within the domain of circular action by connecting two circles to each other [SEE Figure 3]. Thus, while the difference between the arithmetic and geometric clearly presents a change in curvature, the power associated with generating two geometric means, in first approximation, seems to require only another circle, and hence, no change in curvature.

Yet, when the specific physical problem of doubling the cube is posed—that is, to find two geometric means between two determined extremes—the existence of the higher power emerges in the map as a new type of curvature [SEE Figure 4]. As can be seen in the figure, to find two geometric means between OB and OA, we must find a position for point P along the circumference of the circle, such that line OB is one-half OA. This will occur somewhere along the pathway travelled by B, as P moves around the circle from O to A. But, as the dotted line which traces that path indicates, this curve is

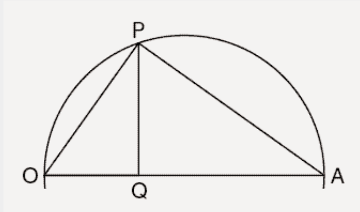

FIGURE 2. *A right angle in a semicircle produces the geometric proportions OQ:OP::OP:OA.*

not circular; in fact, it is non-uniform with respect to the circle. Thus, the existence of the yet-to-be-discovered principle, emerges through the presence of an anomalous change in curvature on our map.

This anomaly takes on an entirely different characteristic in Archytas's construction using the torus, cylinder, and cone [SEE Figure 5]. When the torus and cylinder are generated by rotating one circle (OPA) orthogonally around another (OQD) with point O fixed, the motion of point P is now simultaneously on two different curves: the circle, and the dotted curve formed by the intersection of the torus and the cylinder. An observer facing the rotating circle, who was rotating at exactly the same speed as the circle, would only see point P move around the circumference of the circle, and would adequately conclude that one geometric mean between two extremes is a function of circular action alone. But, as indicated above, the emergence of the non-circular curvature of the path of point B, would indicate to such an observer, the existence of a new principle which causes the motion of P around the circle. Archytas's construction takes that new principle into account, by determining the motion of P around the circle as a function of the motion of P along the curve formed by the intersection of the torus and cylinder. In other words, the circular rotation of P is only a shadow of a higher form of curvature. That latter curve expresses both the power to produce one geometric mean

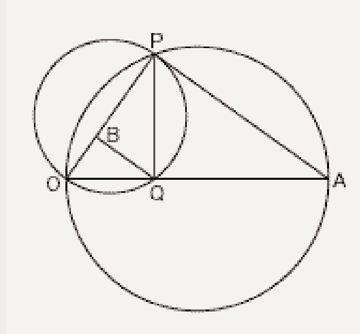

FIGURE 3. *Two circles generate two geometric means between two extremes OB:OQ::OQ:OP::OP:OA.*

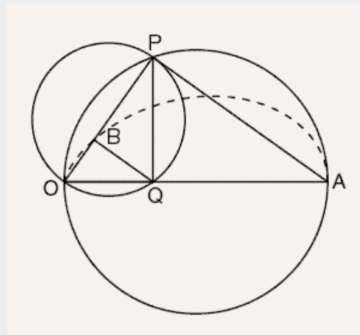

FIGURE 4. *As P moves around the circumference of the semicircle from A to O, B moves on the non-uniform curve (dashed line).*

between two extremes, and also, when combined with a cone, to produce two [SEE Figure 6].

Two other examples, presented summarily, will help illustrate the point. Kepler, like all astronomers before and since, observed the motions of the planets as circular arcs on the inside of a sphere. His discovery of the elliptical nature of these orbits occurred, not by suddenly seeing an ellipse, but by his recognition that the deviation of 8′ of arc between the circular image of the planet's orbit on the celestial sphere, and the circular image of the Earth's motion (as reflected in the motion of the fixed stars on that same celestial sphere), was evidence of a new

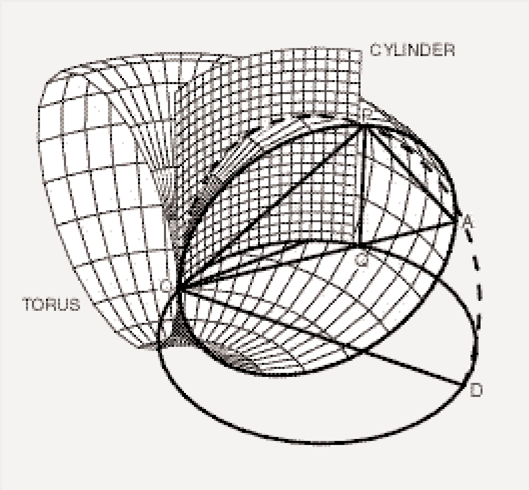

FIGURE 5. *As circle OPA rotates around circle OQD, point P moves on both the circumference of circle OPA and curve OPD (dashed line) formed by the intersection of the torus and cylinder.*

principle of planetary motion. The new principle manifested itself as a change in curvature within his map of principles. He measured that change in curvature by measuring equal areas instead of equal arcs, and measuring eccentricities by the proportions that correspond to musical harmonics.

Similarly, Leibniz and Bernoulli determined that the catenary was not the parabola that Galileo wrongly believed it to be, by showing that the slight deviation of the curvature of the physical hanging chain from the curvature of the parabola, was evidence that the chain was governed by a different principle than the one Galileo assumed. Galileo demanded, as if in a bi-polar rage, that the chain conform to a parabolic shape, because he was obsessed with his mathematical formula that the velocity of a falling body varies according to the square root of the distance fallen. Leibniz and Bernoulli demonstrated that, in truth, the chain was obeying a higher principle, the non-algebraic, transcendental principle associated with Leibniz's discovery of natural logarithms—a principle which the enraged Galileo was incapable of conceiving.

Gaussian Curvature

In order to proceed further, it is important to distinguish between commonplace, sense-certainty notions of curvature, and the rigorous understanding of that idea associated with Gauss. The commonplace notion, associated with the doctrines of Galileo, Newton, Euler, *et al.,* is that curvature is a deviation from the straight. But, from the standpoint of the planet, for example, "straight," is a unique elliptical path; or, from the standpoint of a link in a chain, "straight" is the catenary curve. It is only a self-deluded fool who thinks that "straight" can be determined by some arbitrary, abstract dictate. Rather, "straight" is a function of the set of principles that are determining the action. The addition of a new principle will change the direction of "straight." That change in principle is measured as a change in curvature.

This is the standpoint from which Gauss developed his "General Investigations of Curved Surfaces." He considered a curved surface to be a set of invariant principles which determined the nature of action on that surface. As long as that set of principles was not changed, the nature of the action did not change, even if the surface was bent or stretched. The nature of the action could only be changed, by a change in the set of principles that defined the surface. Gauss measured such a change in principle as a change in curvature, which in turn, determined what is "straight" with respect to that set of principles.

Furthermore, Gauss showed, as Leibniz had done for curves, that this set of invariant principles was expressed in even the smallest elements of the surface. Consequently, the curvature of the surface could be determined from the smallest pieces of "straight" curves (geodesics), and their directions.

The method Gauss developed to measure curvature, had its roots in Kepler's method of measuring the elliptical nature of the planetary orbits, which method was generalized by Leibniz in his development of the calculus. Confronting the difficulty of directly measuring the planet's non-uniform, elliptical motion, Kepler mapped the constantly changing speed and direction of the planet onto a circular path, and was thus able to measure the planet's

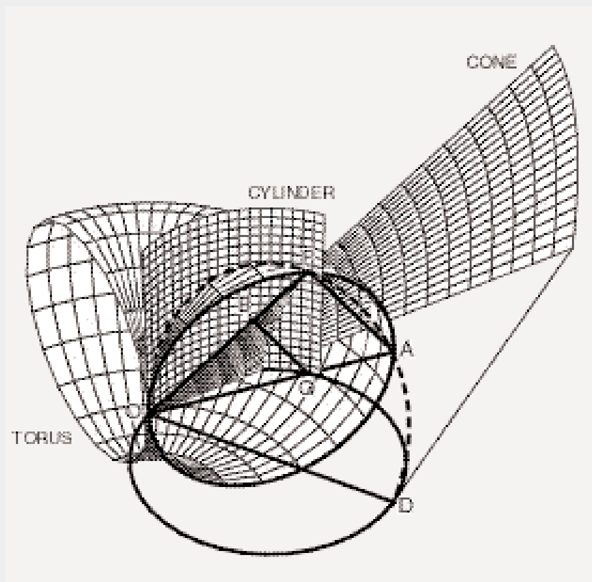

FIGURE 6. *Two geometric means between two extremes can be found by intersecting curve OPD (dashed line) with a cone. Here OB : OQ : : OQ : OP : : OP : OA.*

action according to the relationships among the three anomalies (eccentric, mean, and true) that appeared in the circular map.*

To measure the curvature of a surface, Gauss extended Kepler's method from the mapping of a curve onto a circle, to the mapping of a surface onto a sphere, a method he likened to the ancient use of the celestial sphere in astronomy. In that case, the motion of a celestial body is mapped by the changing directions of lines from the observer, to the body's image on the inside of the celestial sphere. Since whatever principle is governing the body's motion, is governing the changes in direction of those lines, measuring the map of those changes in direction is an indirect measurement of the governing principle.

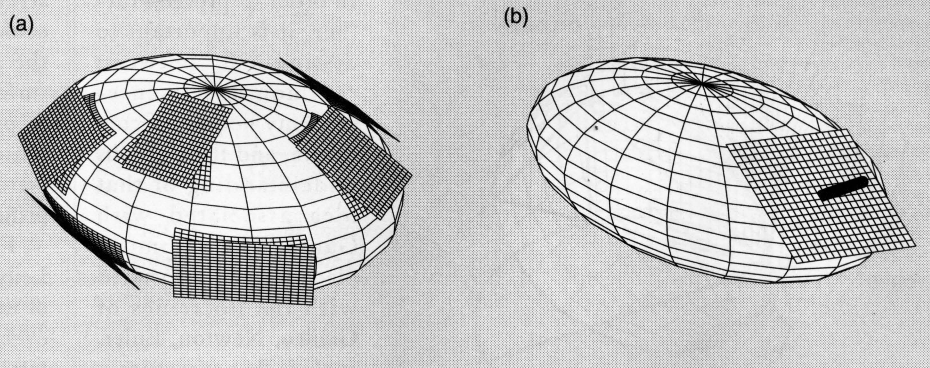

FIGURE 7. *(a) At each point on a surface, there is only one unique tangent plane. (b) At any point on the tangent plane, there is one unique line, called a normal, perpendicular to the tangent plane.*

Gauss recognized that the invariant principles governing a surface could be expressed by the changing direction of the lines perpendicular to the surface at every point, called "normals." While at any point on a surface there are an infinite number of tangents, there is a unique tangent plane at each point, which contains all the tangents; this tangent plane in turn defines a unique normal perpendicular to it [SEE Figure 7]. Thus, the direction of the normal is a function of the curvature of the surface. (This is a principle of physical geometry, as exemplified by the determination of the physical horizon as that direction that is perpendicular to the pull of gravity.)

The sphere has the unique characteristic that all its normals are also radial lines. Using this property, Gauss was able to map every normal to a surface, to a corresponding radial line of a sphere that points in the same direction. As the normal moves around on a surface, its direction changes. If the radial line of the sphere is made to change its direction in the same way as the normal, then the curve it traces out on the surface of the sphere will reflect the principle governing the changes in direction of the normal on the surface.

* See Jonathan Tennenbaum and Bruce Director, "How Gauss Determined the Orbit of Ceres," *Fidelio,* Summer 1998 (Vol. VII, No. 2), pp. 29-34.

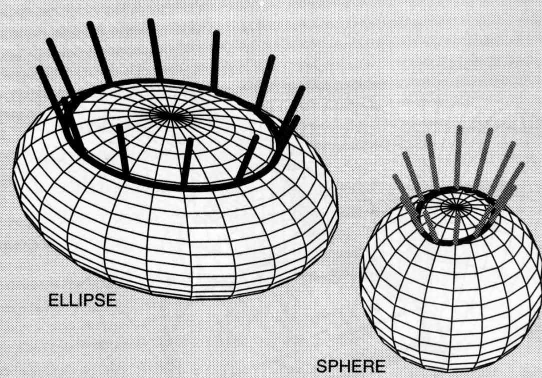

FIGURE 8. *The curvature of the enclosed area of the ellipsoid is mapped by parallel normals to the sphere. Gauss called the area of the spherical map, the "total curvature." In the case shown, a relatively large area of the ellipsoid maps to a relatively small area of the sphere, indicating that this area of the ellipsoid has a small total curvature.*

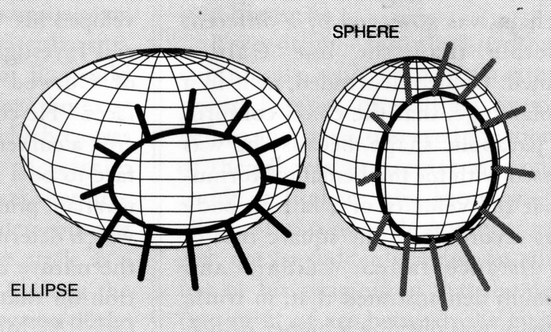

FIGURE 9. *On a different part of the ellipsoid, a relatively smaller enclosed area produces a larger mapping on the sphere, indicating a greater total curvature for this part of the ellipsoid.*

This is illustrated in Figures 8 and 9. In these examples, the part of the ellipsoid marked out by the closed curve, is mapped onto a sphere. As the solid black stick (normal) moves around the ellipsoid, its changing direction is determined by the changing curvature of the surface. These changes are mapped onto a sphere, by the motion of the tinted gray stick, which emanates from the center of the sphere and is always pointing in the same direction as the solid black stick. Gauss called the area marked out by the tinted gray stick on the sphere, the "total or integral curvature" of the surface. If the solid black stick were moving along a plane, its direction would not change, and the tinted gray stick would not move. Since this would obviously mark out no area, Gauss defined a plane as a surface of zero curvature. The greater the area marked out on the sphere, the greater the curvature of the surface being mapped.

This can be seen from the above two examples. In Figure 8, the solid black stick is moving around a large area of the ellipsoid, but because that region is less curved, its direction doesn't change very much, and the corresponding area on the sphere is small. While, in Figure 9, the area on the ellipsoid is small, but very curved, so the area marked out on the sphere is larger.

This total curvature does not change even if the surface is deformed by being bent or stretched. To understand this, try, for example, to determine the spherical map of part of a cone or a cylinder.

Using this method, Gauss was able to not only measure the "amount" of curvature, he was also able to distinguish different types of curvature that are determined by different sets of principles. For example, Figure 10 shows the mapping of a surface called a "monkey saddle." (This type of surface should be familiar to those who have been inspired by previous Pedagogical Exercises to study Gauss's 1799 proof of the "Fundamental Theorem of Algebra.") In this mapping, the curvature of the area denoted by the closed curve on the monkey saddle is mapped onto the sphere. As the solid black stick moves once

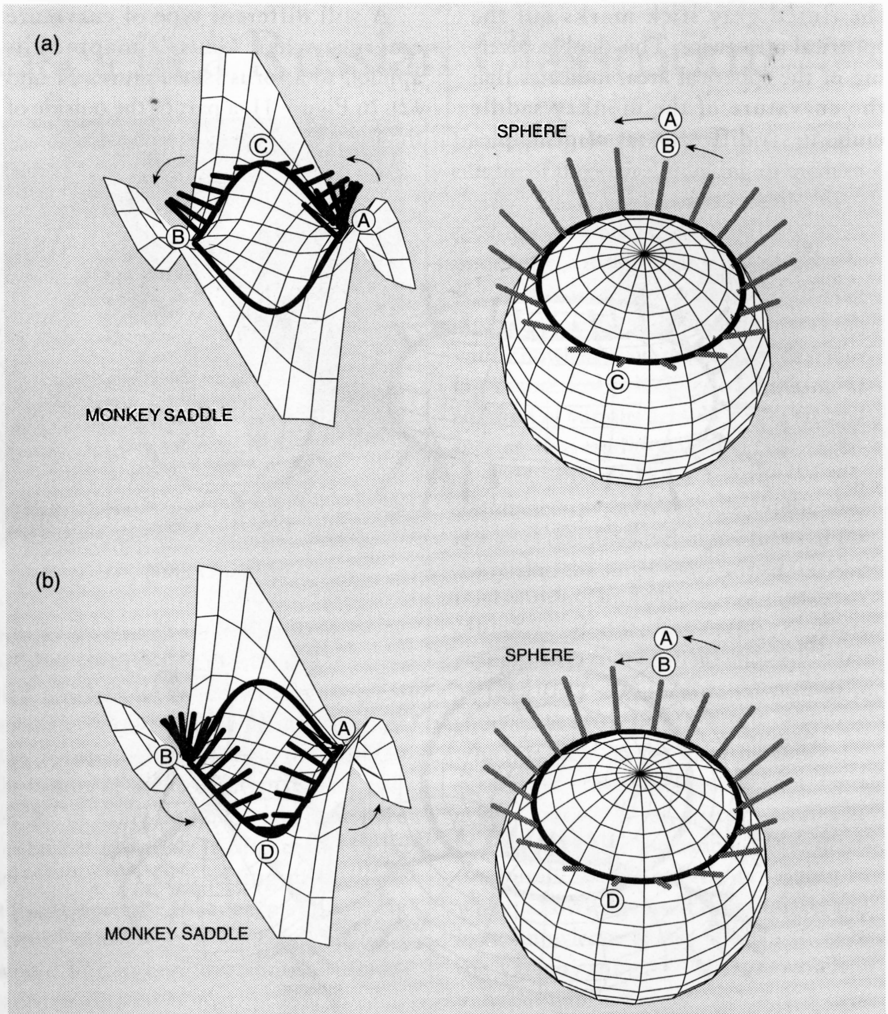

FIGURE 10. *The mapping of the monkey saddle covers the sphere twice. (a) The normals from A to C to B cover only half the enclosing curve on the monkey saddle, but make one full rotation on the sphere. (b) A second rotation is produced by the mapping of the normals from B to D back to A.*

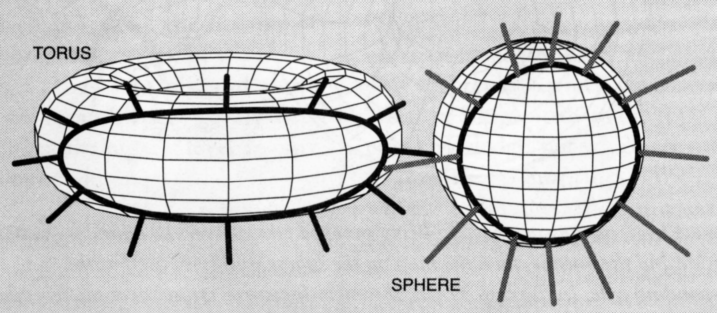

FIGURE 11. *The outside part of a torus maps to a corresponding area on the sphere.*

around the area on the monkey saddle, the tinted gray stick marks out the spherical area twice. This double covering of the spherical area, indicates that the curvature of the monkey saddle embodies a different set of principles than the curvature of the ellipsoid.

A still different type of curvature emerges when Gauss's mapping is applied to a torus [SEE Figures 11 and 12]. In Figure 11, a part of the outside of the torus is mapped, producing a corresponding area on the sphere, similar to what happened in the case of the ellipsoid. But, in Figure 12, the area of the torus is situated on both its inner and outer parts. The mapping of these directions produces a "figure-eight" type of curve on the sphere, which crosses itself at both the north and south poles. Each time the solid black stick crosses the circle that forms the boundary between the inner and outer parts of the torus, the tinted gray stick crosses one of the poles of the sphere, with two loops of the double figure-eight corresponding to the inner part of torus, and the center loop to the outer part. Thus, the area on the torus is bounded by a non-intersecting curve, while its map on the sphere is bounded by an intersecting one. The presence of this singularity on the spherical map indicates that the boundary between the inner and outer parts of the torus is a transition from one type of curvature to another. Consequently, the torus must be governed by a different set of principles than either the ellipsoid or the monkey saddle—a set of principles which includes a transition between two different types of curvature.

To summarize: For the ellipsoid, the Gaussian mapping produces a simple area whose size varies with the curvature of the surface. The mapping of the monkey saddle produces an area that is double-covered. The mapping of the torus produces two singularities, one on the top boundary between the inside and the outside of the torus, and the other at the bottom boundary. These mappings not only measure the "amount" of total curvature of the part of the surface mapped, but the appearance of anomalies and singularities in the mapping indicate the presence of *additional* principles of curvature as well.

Like the character Chorus in Shakespeare's *Henry V,* who, alone on an empty stage, summons the imagination of the audience to envision the real principles of history and statecraft that are to be depicted, these anomalies and singularities call the attention of the scientist to imagine the set of principles which produced them. That is where real history, and science, are made.

—Bruce Director

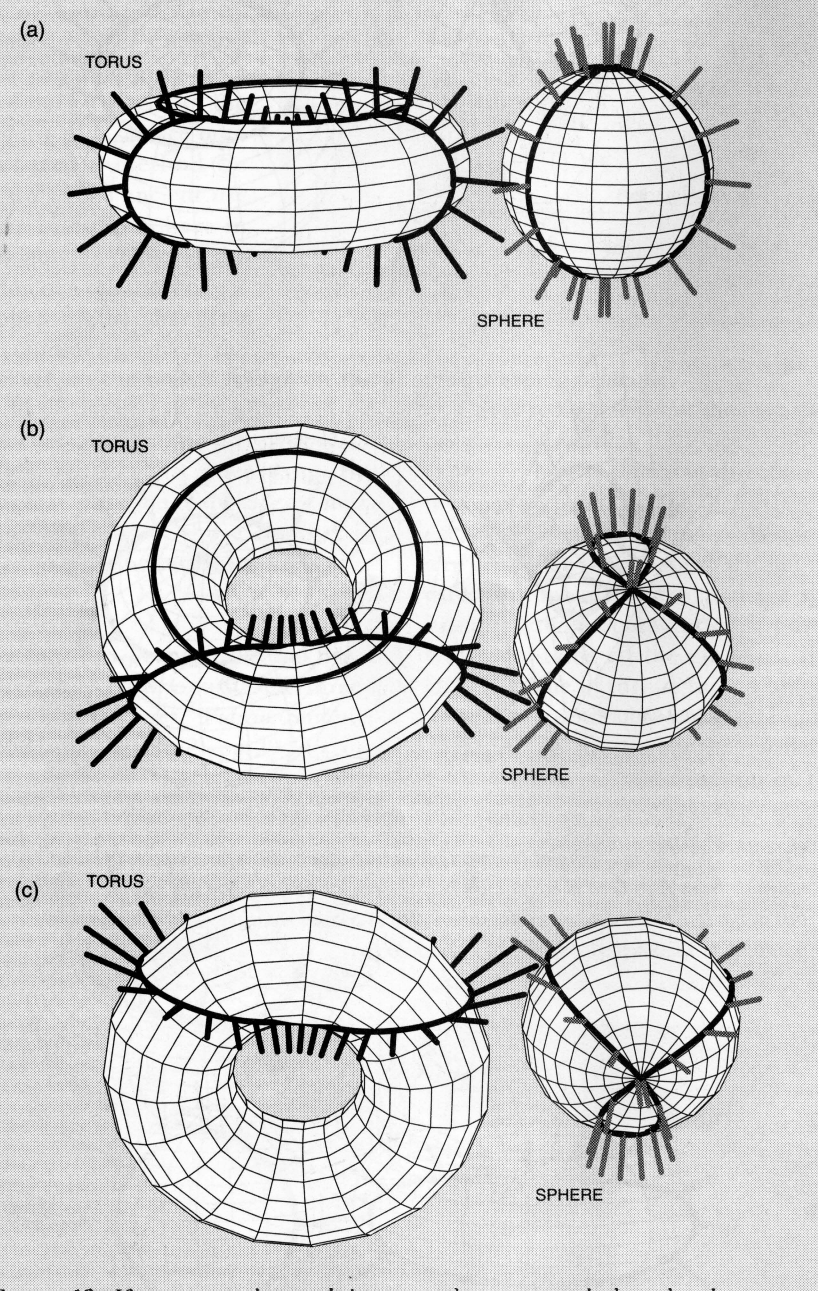

FIGURE 12. *If, on a torus, the area being mapped crosses over the boundary between the inside and the outside, then the map on the sphere will cross itself at the corresponding pole. In the case shown, the enclosing curve crosses over the boundary twice, on both the top and the bottom of the torus. (a) Side view. (b) Top view. (c) Bottom view.*