Chapter 6

Uniting Beauty and Necessity

A great crisis and a great opportunity were created by Giuseppe Piazzi's startling observations of a new object in the sky, in the early days of 1801. Astronomers were now forced to confront the problem of determining the orbit of a planet from only a few observations. Before Piazzi's discovery, C.F. Gauss had considered this problem purely for its intellectual beauty, although anticipating its eventual practical necessity. Others, mired in purely practical considerations, ignored Beauty's call, only to be caught wide-eyed and scrambling when presented with the news from Piazzi's observatory in Palermo. Gauss alone had the capacity to unite Beauty with Necessity, lest humanity lose sight of the newly expanded Universe.

As we continue along the circuitous path to rediscovering Gauss's method for determining the orbit of Ceres, we are compelled to linger a little longer at the beginning of an earlier century, when a great crisis and opportunity arose in the mind of someone courageous and moral enough to recognize its existence. In those early years of the Seventeenth century, as Europe disintegrated into the abyss of the Thirty Years War, Johannes Kepler's quest for beauty led him to the discoveries that anticipated the crisis Gauss would later face, and laid the groundwork for its ultimate solution.

In the last chapter, we retraced the first part of Kepler's great discoveries: that the time which a planet takes to pass from one position of its orbit to another, is proportional to the area of the sector formed by the lines joining each of those two planetary positions with the sun, and the arc of the orbit between the two points.* But, this discovery of Kepler was immediately thrown into crisis when he compared his calculations to the observed positions of Mars, and the time elapsed between those observations. This combination of the change in the observed position and the time elapsed, is a reflection of the *curvature* of the orbit. Kepler had assumed that the planets orbited the sun in eccentric circles. If, however, the planet were moving on an arc that is not circular, it could be observed in the same positions, but the elapsed time between observations would be different than if it were moving on an eccentric circle. When Kepler calculated his new principle using different observations of the planet Mars, the results were not consistent with a circular planetary orbit.

Kepler's Account

The following extracts from Kepler's *New Astronomy* trace his thinking as he discovers his next principle. Uniquely, Kepler left us with a subjective account of his discovery. Speaking across the centuries, Kepler provides an important lesson for today's "Baby Boomers," who, so lacking the *agapē* to face a problem and discover a creative solution, desperately need the benefit of Kepler's honest discussion of his own mental struggle.

> You see, my thoughtful and intelligent reader, that the opinion of a perfect eccentric circle drags many incredible things into physical theories. This is not, indeed, because it makes the solar diameter an indicator for the planetary mind, for this opinion will perhaps turn out to be closest to the truth, but because it ascribes incredible facilities to the mover, both mental and animal.
>
> Although our theories are not yet complete and perfect, they are nearly so, and in particular are suitable for the motion of the sun, so we shall pass on to quantitative consideration.

It was in the "nearly so," the infinitesimal, that Kepler's crisis arose. He continues, a few chapters later:

> You have just seen, reader, that we have to start anew. For you can perceive that three eccentric positions of Mars and the same number of distances from the sun, when the law of the circle is applied to them, reject the aphelion found above (with little uncertainty). This is the source of our suspicion that the planet's path is not a circle.

Having come to the realization that he must abandon the hypothesis of circular orbits, he first considers ovals.

> Clearly, then, the orbit of the planet is not a circle, but comes in gradually on both sides and returns again to the circle's distance at perigee. One is accustomed to call the shape of this sort of path "oval."

Yet, after much work, Kepler had to admit that this too was incorrect:

> When I was first informed in this manner by [Tycho] Brahe's most certain observations that the orbit of the planet is not exactly circular, but is deficient at the sides, I judged

* This principle has now become known as Kepler's Second Law, even though it was the first of Kepler's so-called three laws to be discovered. Kepler never categorized his discoveries of principles into a numbered series of laws. The codification of Kepler's discovery, to fit academically acceptable Aristotelean categories, has masked the true nature of Kepler's discovery and undermined the ability of others to know Kepler's principles, by rediscovering them for themselves.

that I also knew the natural cause of the deflection from its footprints. For I had worked very hard on that subject in Chapter 39. . . . In that chapter I ascribed the cause of the eccentricity to a certain power which is in the body of the planet. It therefore follows that the cause of this deflecting from the eccentric circle should also be ascribed to the same body of the planet. But then what they say in the proverb—"A hasty dog bears blind pups"—happened to me. For, in Chapter 39, I worked very energetically on the question of why I could not give a sufficiently probable cause for a perfect circle's resulting from the orbit of a planet, as some absurdities would always have to be attributed to the power which has its seat in the planet's body. Now, having seen from the observations that the planet's orbit is not perfectly circular, I immediately succumbed to this great persuasive impetus. . . .

Self-consciously describing the emotions involved:

> And we, good reader, can fairly indulge in so splendid a triumph for a little while (for the following five chapters, that is), repressing the rumors of renewed rebellion, lest its splendor die before we shall go through it in the proper time and order. You are merry indeed now, but I was straining and gnashing my teeth.

And, continuing:

> While I am thus celebrating a triumph over the motions of Mars, and fetter him in the prison of tables and the leg-irons of eccentric equations, considering him utterly defeated, it is announced again in various places that the victory is futile, and war is breaking out again with full force. For while the enemy was in the house as a captive, and hence lightly esteemed, he burst all the chains of the equations and broke out of the prison of the tables. That is, no method administered geometrically under the direction of the opinion of Chapter 45 was able to emulate in numerical accuracy the vicarious hypotheses of Chapter 16 (which has true equations derived from false causes). Outdoors, meanwhile, spies positioned throughout the whole circuit of the eccentric—I mean the true distances—have overthrown my entire supply of physical causes called forth from Chapter 45, and have shaken off their yoke, retaking their liberty. And now there is not much to prevent the fugitive enemy's joining forces with his fellow rebels and reducing me to desperation, unless I send new reinforcements of physical reasoning in a hurry to the scattered troops and old stragglers, and, informed with all diligence, stick to the trail without delay in the direction whither the captive has fled. In the following chapters, I shall be telling of both these campaigns in the order in which they were waged.

In another place, Kepler writes:

> *"Galatea seeks me mischievously, the lusty wench,*
> *She flees the willows, but hopes I'll see her first."*
>
> It is perfectly fitting that I borrow Virgil's voice to sing this about Nature. For the closer the approach to her, the more petulant her games become, and the more she again and again sneaks out of the seeker's grasp, just when he is about to seize her through some circuitous route. Nevertheless, she never ceases to invite me to seize her, as though delighting in my mistakes.
>
> Throughout this entire work, my aim has been to find a physical hypothesis that not only will produce distances in agreement with those observed, but also, and at the same time, sound equations, which hitherto we have been driven to borrow from the vicarious hypothesis of Chapter 16. . . .

And, after much work, he finally arrives at the answer the Universe has been telling him all along:

> The greatest scruple by far, however, was that, despite my considering and searching about almost to the point of insanity, I could not discover why the planet, to which a reciprocation LE on the diameter LK was attributed with such probability, and by so perfect an agreement with the observed distances, would rather follow an elliptical path, as shown by the equations. O ridiculous me! To think that reciprocation on the diameter could not be the way to the ellipse! So it came to me as no small revelation that through the reciprocation an ellipse was generated. . . .

With the discovery of an additional principle, Kepler has accomplished the next crucial step along the road Gauss would later extend by the determination of the orbit of Ceres. The discovery that the shape of the orbit of the planet Mars (later generalized to all planets) was an ellipse, would be later generalized even further to include all conic sections, when other heavenly bodies, such as comets, were taken into account.

But now a new crisis developed for Kepler. What we discussed in the last chapter—the elegant way of calculating the area of the orbital sector, which is proportional to the elapsed time—no longer works for an ellipse. For that method was discovered when Kepler was still assuming the shape of the planet's orbit to be a circle.

To grasp this distinction, the reader will have to make the following drawings:

First re-draw Figure 5.5. **(Figure 6.1)** [For the reader's convenience, figures from previous chapters are displayed again when re-introduced.]

The determination of the area formed by the motion of the planet in a given interval of time, was defined as the "sum" of the infinite number of radial lines obtained as the planet moves from P_1 to P_2. This "sum," which Kepler represents by the area AP_1P_2, is calculated by subtracting the area of the triangle ABP_2 from the circular sector BP_1P_2. But, as noted previously, determining the area of triangle ABP_2 depended on the sine of the angle ABP_2, i.e., P_2N, which Kepler, as a student of Cusa, recognized was transcendental to the arc P_1P_2, thus making

FIGURE 6.1. *Kepler's method of calculating swept-out areas for an eccentric circular orbit.*

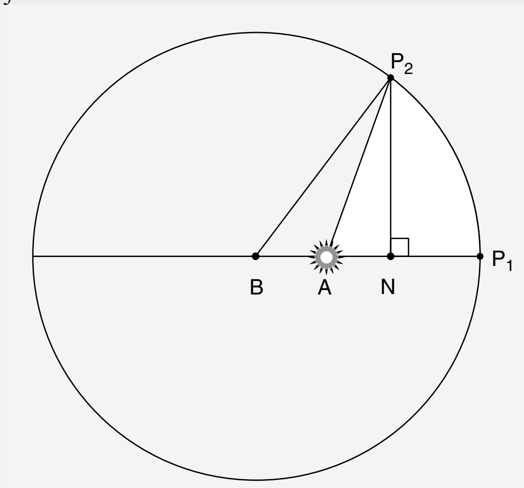
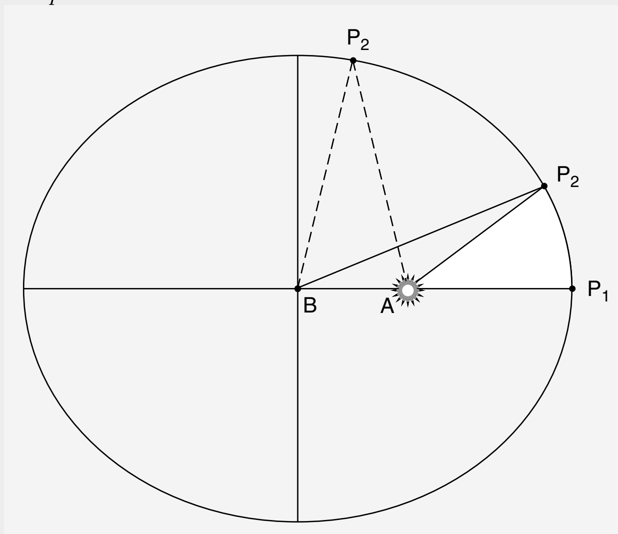


FIGURE 6.2. *Kepler's elliptical orbit hypothesis. Here, length P_2B is not constant, but constantly changing at a changing rate. What lawful process now underlies the generation of swept-out areas?*



a direct algebraic calculation impossible.

But now that Kepler has abandoned the circular orbit for an elliptical one, this problem is compounded. For the circular arc is characterized by constant uniform curvature, while the curvature of the ellipse is non-uniform, constantly changing. Thus, if we abandon the circular orbit and accept the elliptical one, as reality demands, the simplicity of the method for determining the area of the orbital sector disappears.

A Dilemma, and a Solution

What a dilemma! Our Reason, following Kepler, leads us to the hypothesis that the area of the orbital sector swept out by the planet, is proportional to the time it takes for the planet to move through that section of its orbit. But, following Kepler, our Reason, guided by the actual observations of planetary orbits, also leads us to abandon the circular shape of the orbit, in favor of the ellipse, and to lose the elegant means for applying the first discovery.

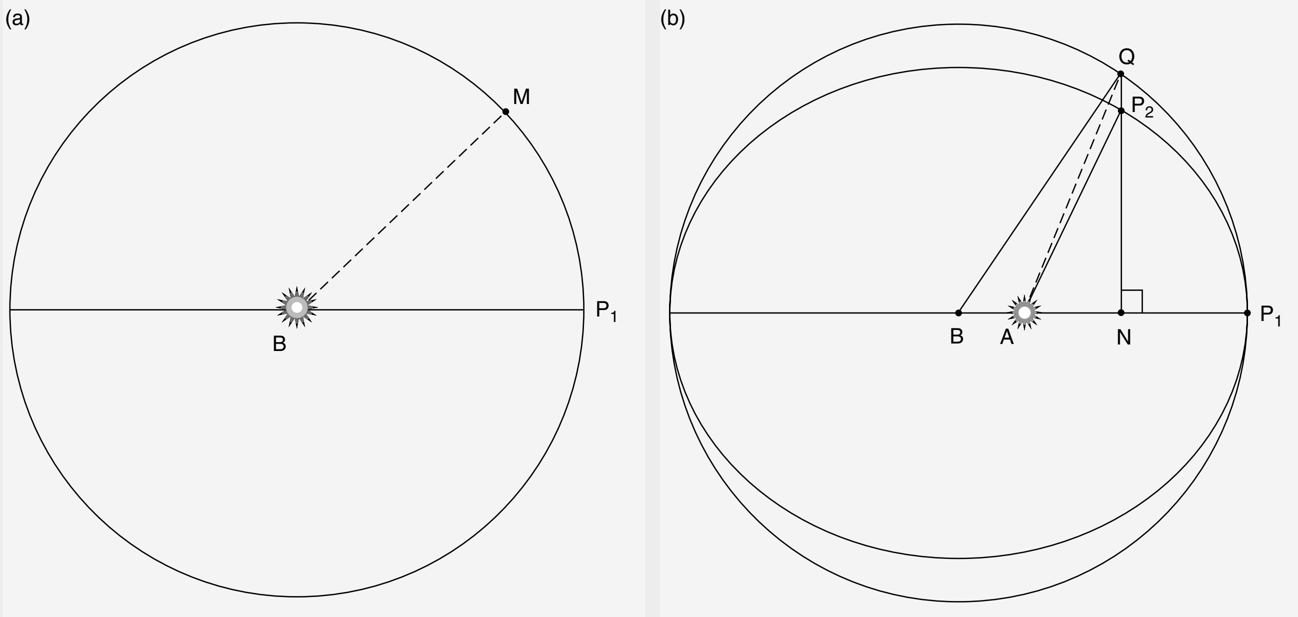
This is no time to emulate Hamlet. Our only way out is to forge ahead to new discoveries. As has been the case so far, Kepler does not let us down.

For the next step, the reader will have to draw another diagram. **(Figure 6.2)** This time draw an ellipse, and call the center of the ellipse B and the focus to the right of the center A. Call the point where the major axis intersects the circumference of the ellipse closest to A, point P_1. Mark another point on the circumference of the ellipse (moving counter-clockwise from P_1), point P_2. As in the previous diagram, A represents the position of the sun, P_1 and P_2 represent positions of the planet at two different points in time, and the circumference of the ellipse represents the orbital path of the planet.

Now compare the shape of the orbital sector in the two different orbital paths, circular and elliptical, as shown in Figures 6.1 and 6.2. The difference in the *type* of curvature between the two is reflected in the *type* of change in triangle ABP_2 as the position P_2 changes. In the circular orbit, the length of line P_2A changes, but the length of line P_2B, being a radius of the circle, remains the same. In the elliptical orbit, the length of the line P_2B also changes. In fact, the rate of change of the length of line P_2B is itself constantly changing.

To solve this problem, Kepler discovers the following relationship. Draw a circle around the ellipse, with the center at B and the radius equal to the semi-major axis. **(Figure 6.3b)** This circle circumscribes the ellipse, touching it at the aphelion and perihelion points of the orbit. Now draw a perpendicular from P_2 to the major axis, striking that axis at a point N, and extend the perpendicular outward until it intersects the circle, at some point Q. Recall one of the characteristics of the ellipse (Figure 1.7b): An ellipse results from "contracting" the circle in the direction perpendicular to the major axis according to some fixed ratio. In other words, the ratio $NP_2 : NQ$ has the same constant value for all positions of P_2. Or, said inversely, the circle results from "stretching" the ellipse outward from the major axis by a certain constant factor, as if on a pulled rubber sheet. It is easy to see that the value of that factor must be the ratio of the major to minor axes of the ellipse.

FIGURE 6.3. *Ironies of Keplerian motion. (a) M is the position a planet would reach after a given elapsed time, assuming it started at P_1 and travelled on the circular orbit with the sun at the center B. (b) P is the corresponding position on the elliptical orbit with the sun at the focus A. The orbital period is the same as (a), but the arc lengths travelled vary with the changing distance of the planet from the sun (Kepler's "area law"). Q is the position the planet would reach if it were moving on the circle, but with the sun at A rather than the center B. For equal times, the area P_1MB will be equal to area P_1QA, the latter being in a constant ratio to the area P_1P_2A.*



With a bit of thought, it might occur to us that the result of such "stretching" will be to change all *areas* in the figure by the same factor. Look at Figure 6.3b from that standpoint. What happens to the *elliptic sector* which we are interested in, namely P_1P_2A, when we stretch out the ellipse in the indicated fashion? It turns into the *circular sector* P_1QA! Accordingly, the area of the elliptical sector swept out by P_2, and that swept out on the circle by Q, stay in a constant ratio to each other throughout the motion of P_2. Since the planet (or rather, the radial line AP_2) sweeps out equal areas on the ellipse in equal times, in accordance with Kepler's "area law," the corresponding point Q (and radial line AQ) will do the same thing on the circle.

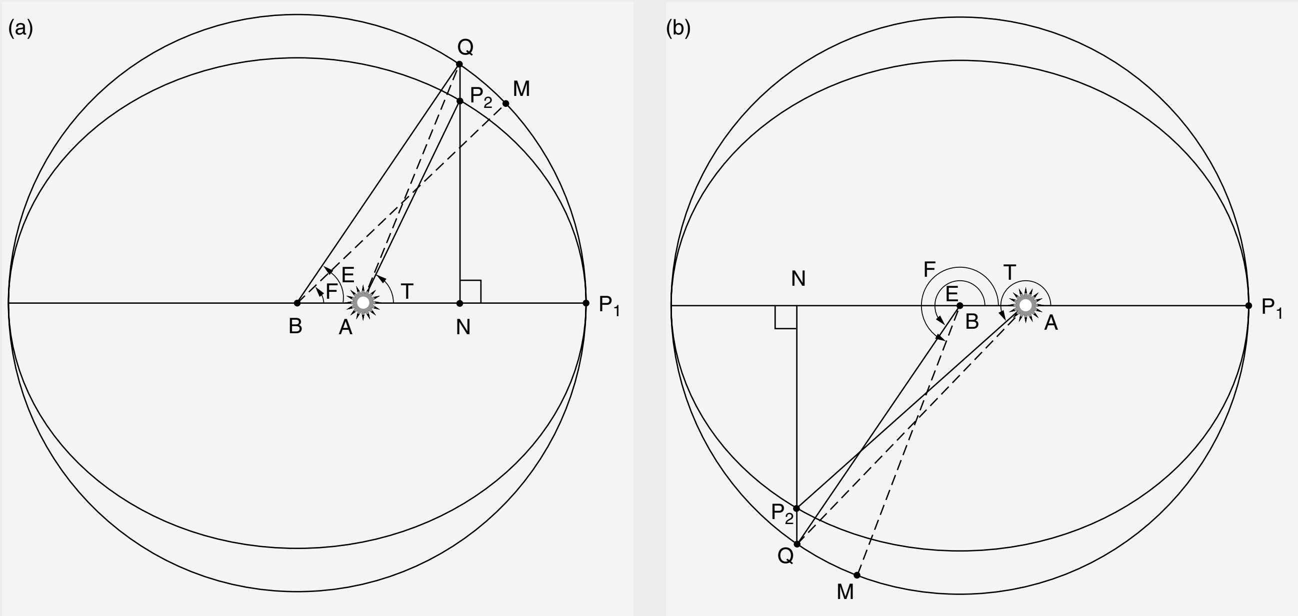
This crucial insight by Kepler unlocks the whole problem. First, it shows that Q is just the position which the planet would occupy, were it moving on an eccentric-circular orbit in accordance with the "area law," as Kepler had originally believed. The difference in position between Q and the actual position P_2 (as observed, for example, from the sun) reflects the non-circular nature of the actual orbit. Second, the constant proportionality of the swept-out areas permits Kepler to reduce the problem of calculating the motion on the ellipse, to that of the eccentric circle, whose solution he has already obtained. (SEE Chapter 5)

Further details of Kepler's calculations need not concern us here. What is most important to recognize, is the *triple* nature of the deviation of a real planet's motion from the hypothetical case of perfect circular motion with the sun at the center—a deviation which Kepler measured in terms of three special angles, called "*anomalies*." First, the sun is not at the center. Second, the orbit is not circular, but elliptical. Third, the speed of the planet varies, depending upon the planet's distance from the sun. For which reason, Kepler's approach implies reconceptualizing, from a higher standpoint, what we mean by the "curvature" of the orbit. Rather than being thought of merely as a geometrical "shape," on which the planet's motion appears to be non-uniform, the "curvature" must instead be conceived of as the motion of the planet moving along the curve *in time*—that is, we must introduce a new conception of *physical space-time*.

In a purely circular orbit, the uniformity of the planet's spatial and temporal motions coincide. That is, the planet sweeps out equal arcs and equal areas in equal times as it moves. Such motion can be completely represented by a single angular measurement.

In true elliptical orbits, however, the motion of the planet can only be completely described by a combination of three angular measurements, which are the three anomalies described below. The uniformity of the "curvature" of the planet's motion finds expression in Kepler's equal-area principle, from the more advanced *physical space-time* standpoint.

FIGURE 6.4. *Non-uniform motion in an elliptical orbit is characterized by the "polyphonic" relationship between the "eccentric anomaly" (angle E), "true anomaly" (angle T), and "mean anomaly" (angle F). (a) As the planet moves from perihelion to aphelion, the true anomaly is greater than the eccentric, which is greater than the mean. (b) After the planet passes aphelion, these relationships are reversed.*



Kepler's conception follows directly from the approach to experimental physics established by his philosophical mentor Nicolaus of Cusa. This may rankle the modern reader, whose thinking has been shaped by Immanuel Kant's neo-Aristotelean conceptions of space and time. Kant considered three-dimensional "Euclidean" space, and a linear extension of time, to be a true reflection of reality. Gauss rejected Kant's view, calling it an illusion, and insisting instead that the true nature of space-time can not be assumed *a priori* from purely mathematical considerations, but must be determined from the physical reality of the Universe.

Kepler's Three Anomalies

The first anomaly is the angle formed by a line drawn from the sun to the planet, and the line of apsides (P_2AP_1 in Figure 6.3b). Kepler called this angle the "equated anomaly." In Gauss's time it was called the "true anomaly." The true anomaly measures the true displacement along the elliptical orbit. The next two anomalies can be considered as two different "projections," so to speak, of the true anomaly.

The second anomaly, called the "eccentric anomaly," is the angle QBP_1, which measures the area swept out had the planet moved on a circular arc, rather than an elliptical one. Since this area is proportional to the time elapsed, it is also proportional, although obviously not equal, to the true orbital sector swept out by the planet.

The third anomaly, called the "mean anomaly," corresponds to the elapsed time, as measured either by area AP_1P_2 or by AP_1Q. It can be usefully represented by the position and angle F at B formed by an *imaginary point M* moving on the circle, whose motion is that which a hypothetical planet would have, if its orbit were the circle and if the sun were at center B rather than A! **(Figure 6.4)** As a consequence of Kepler's Third Law, the total period of the imaginary orbit of M, will coincide with that of the real planet. Hence, if M is taken to be "synchronized" in such a way that the positions of M and the actual planet coincide at the perihelion point P_1, then M and the planet will return to that same point simultaneously after having completed one full orbital cycle.

Kepler established a relationship between the mean and eccentric anomalies, such that, given the eccentric, the mean can be approximately calculated. The inverse problem—that is, given the time elapsed, to calculate the eccentric anomaly—proved much more difficult, and formed part of the considerations provoking G.W. Leibniz to develop the calculus.

The relationship among these three anomalies is a reflection of the *curvature* of space-time relevant to the harmonic motion of the planet's orbit, just as the catenary function described in Chapter 4, reflects such a physical principle in the gravitational field of the Earth. This threefold relationship is one of the earliest examples of what Gauss and Bernhard Riemann would later develop into *hypergeometric,* or *modular functions*—functions in which several seemingly incommensurable cycles are unified into a One.

Kepler describes the relationship between these anomalies this way (we have changed Kepler's labelling to correspond to our diagram):

> The terms "mean anomaly," "eccentric anomaly," and "equated anomaly" will be more peculiar to me. The mean anomaly is the time, arbitrarily designated, and its measure, the area P_1QA. The eccentric anomaly is the planet's path from apogee, that is, the arc of the ellipse P_1P_2, and the arc P_1Q which defines it. The equated anomaly is the apparent magnitude of the arc P_1Q as viewed from A, that is, the angle P_1AP_2.

All three anomalies are zero at perihelion. As the planet moves toward aphelion, all three anomalies increase, with the true always being greater than the eccentric, which in turn is always greater than the mean. At aphelion, all three come together again, equaling 180°. As the planet moves back to perihelion, this is reversed, with the mean being greater than the eccentric, which in turn is greater than the true, until all three come back together again at the perihelion.

Suffice it to say, for now, that Gauss's ability to "read between the anomalies," so to speak, was a crucial part of his ability to hear the new polyphonies sounded by Piazzi's discovery—the unheard polyphonies that the ancient Greeks called the "music of the spheres."

—*BD*

CHAPTER 7

Kepler's 'Harmonic Ordering' Of the Solar System

At this point in our journey toward Gauss's determination of the orbit of Ceres, before plunging into the thick of the problem, it will be worthwhile to look ahead a bit, and to take note of a crucial *irony* embedded in Gauss's use of a generalized form of Kepler's "Three Laws" for the motion of heavenly bodies in conic-section orbits.

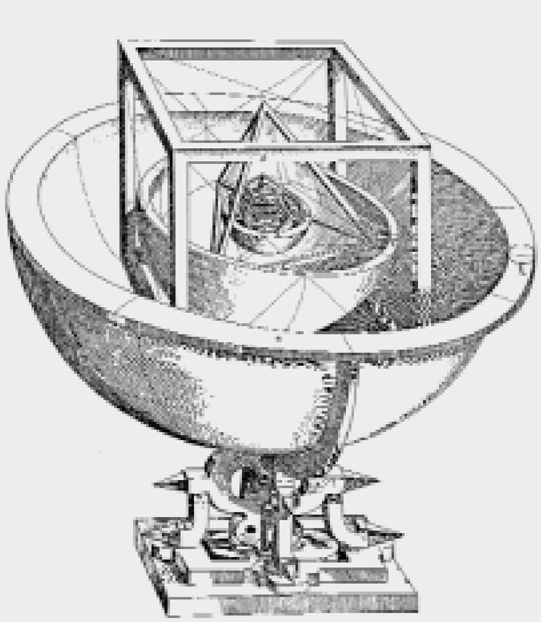


FIGURE 7.1 *(a) Kepler's "harmonic ordering" of the solar system. The planetary orbits are nested according to the ratios of inscribed and circumscribed Platonic (regular) solids, in this model from the "Mysterium Cosmographicum."*

On the one hand, we have the *harmonic ordering* of the solar system as a whole, whose essential idea is put forward by Plato in the *Timaeus,* and demonstrated by Kepler in detail in his *Mysterium Cosmographicum* (*Cosmographic Mystery*) and *Harmonice Mundi (The Harmony of the World).* **(Figure 7.1a)** A crucial feature of that ordering, already noted by Kepler, is the existence of a singular, "dissonant" orbital region, located between Mars and Jupiter—a feature whose decisive confirmation was first made possible by Gauss's determination of the orbit of Ceres. **(Figure 7.1b)**

Although Kepler's work in this direction is incomplete in several respects, that harmonic ordering *in principle* determines not only which orbits or arrays of planetary orbits are possible, but also the *physical characteristics* of the planets to be found in the various orbits. Thus, the Keplerian ordering of the solar system is not only *analogous* to Mendeleyev's natural system of the chemical elements, but ultimately expresses the *same* underlying curvature of the Universe, manifested in the astrophysical and microphysical scales.*

On the other hand, we have Kepler's constraints for the motion of the planets within their orbits, developed step-by-step in the course of his *New Astronomy* (1609), *Harmony of the World* (1619), and *Epitome Astronomiae Copernicanae (Epitome of Copernican Astronomy)* (1621).

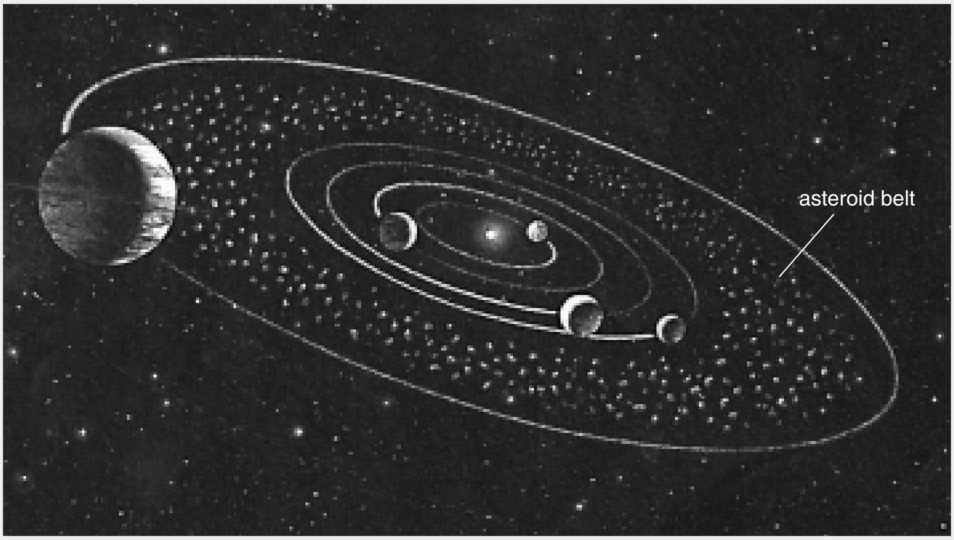


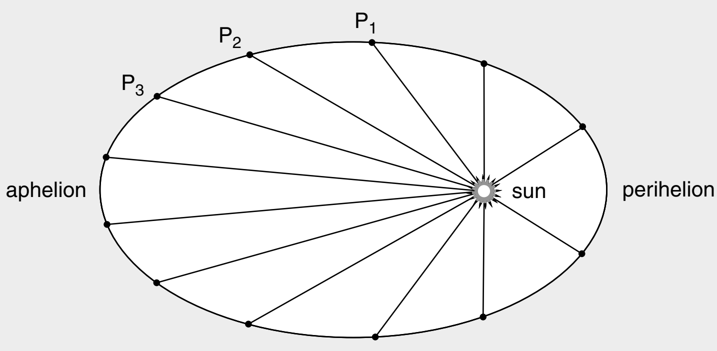
FIGURE 7.1(*b*) *Kepler's model defines a "dissonant" interval, the orbital region between Mars and Jupiter. Decisive confirmation of Kepler's hypothesis was first made possible by Gauss's determination of the orbit of the asteroid Ceres. This region, known today as the "asteroid belt," marks the division between the "inner" and "outer" planets of the solar system, and may be the location of an exploded planet unable to survive at this harmonically unstable position. (Artist's rendering)*

These constraints provide the basis for calculating, to a very high degree of precision, the position and motion of a planet or other object at any time, once the basic spatial parameters of the orbit itself (the "elements" described in Chapter 2) have been determined. The three constraints go as follows.

1. **The area of the curvilinear region, swept out by the radial line connecting the centers of the given planet and the sun, as the planet passes from any position in its orbit to another, is proportional in magnitude to the time elapsed during that motion.** Or, to put it another way: If P_1, P_2, and P_3 are three successive positions of the planet, then the ratio of the area, swept out in going from P_1 to P_2, to the area, swept out in passing from P_2 to P_3, is equal to the ratio of the corresponding elapsed times. **(Figure 7.2)**
2. **The planetary orbits have the form of perfect ellipses, with the center of the sun as a common focus.**
3. **The periodic times of the planets (i.e., the times required to complete the corresponding orbital cycles), are related to the major axes of the orbits in such a way, that the ratio of the *squares* of the periodic times of any two planets, is equal to the ratio of the *cubes* of the corresponding semi-major axes of the orbits.** (The "semi-major axis" is half of the longest axis of the ellipse, or the distance from the center of the ellipse to either of the two extremes, located at the perihelion and aphelion points; for a circular orbit, this is the same as the radius.) Using the semi-major axis and periodic time of the Earth as units, the stated proposition amounts to saying, that the planet's periodic time T, and the semi-major axis

* Lyndon LaRouche has shed light on that relationship, through his hypothesis on the historical generation of the elements—and, ultimately, of the planets themselves—by "polarized" fusion reactions within a Keplerian-ordered, magnetohydrodynamic plasmoid, "driven" by the rotational action of the sun. In that process, Mendeleyev's harmonic values for the chemical elements, and the congruent, harmonic array of orbital corridors of the planets, *predate* the generation of the elements and planets themselves!

FIGURE 7.2. *Kepler's constraint for motion on an elliptical orbit. The ratios of elapsed times are proportional to the ratios of swept-out areas. In equal time intervals, therefore, the areas of the curvilinear sectors swept out by the planet, will be equal—even though the curvilinear distances traversed on the orbit are constantly changing. In the region about perihelion, nearest the sun, the planet moves fastest, covering the greatest orbital distance; whereas, at aphelion, farthest from the sun, it moves most slowly, covering the least distance. This constraint is known as Kepler's "area law," later referred to as his "Second Law."*



A, of the planet are connected by the relation:

$$T \times T = A \times A \times A.$$

(So, for example, the semi-major axis of Mars' orbit is very nearly 1.523674 times that of the Earth, while the Mars "year" is 1.88078 Earth years.) **(Table I)**

In the next chapter, we shall present Gauss's generalized form of these constraints, applied to hyperbolic and parabolic, as well as to elliptical, orbits.

Unfortunately, in the context of ensuing epistemological warfare, Kepler's constraints were ripped out of the pages of his works, severing their intimate connection with the harmonic ordering of the solar system as a whole, and finally dubbed "Kepler's Three Laws." The resulting "laws," taken in and of themselves, do not specify which orbits are possible, nor which actually occur, might have occurred, or might occur in the future; nor do they say anything about the character of the planet or other object occupying a given orbit.

This flaw did not arise from any error in Kepler's work *per se*, but was imposed from the outside. Newton greatly aggravated the problem, when he "inverted" Kepler's constraints, to obtain his "inverse square law" of gravitation, and above all when he chose—for political reasons—to make that "inversion" a vehicle for promoting a radical-empiricist, Sarpian conception of a Universe governed by pair-wise interactions in "empty" space.

However, apart from the distortions introduced by Newton *et al.,* there does exist a paradoxical relationship—of which Gauss was clearly aware—between the three constraints, stated above, and Kepler's harmonic ordering of the solar system as a whole. While rejecting the notion of Newtonian pair-wise interactions as elementary, we could hardly accept the proposition, that the orbit and motion of any planet, does not reflect the rest of the solar system in some way, and in particular the existence and motions of all the other planets, within any arbitrarily small interval of action. Yet, the three constraints make no provision for such a relationship! Although Kepler's constraints are approximately correct within a "corridor" occupied by the orbit, they do not account for the "fine structure" of the orbit, nor for certain other characteristics which we know must exist, in view of the ordering of the solar system as a whole.

A New Physical Principle

Hence the irony of Gauss's approach, which applies Kepler's three constraints as the basis for his mathematical determination of the orbit of Ceres from three observations, *as a crucial step toward uncovering a new physical principle which must manifest itself in a discrepancy, however "infinitesimally small" it might be, between the real motion, and that projected by those same constraints!*

Compare this with the way Wilhelm Weber later derived his electrodynamic law, and the necessary existence of a "quantum" discontinuity on the microscopic scale. Compare this, more generally, with the method of "modular arithmetic," elaborated by Gauss as the basis of his *Disquisitiones Arithmeticae.* Might we not consider any given hypothesis or set of physical principles, or the corresponding functional "hypersurface," as a "modulus," relative to which we are concerned to define and measure various species of discrepancy or "remainder" of the real process, that in turn express the effect of a new physical principle? Thus, we must discriminate, between arrays of phenomena which are "similar," or congruent, in the sense of relative agreement with an existing set of principles, and the species of anomaly we are looking for.

The concept of a series of successive "moduli" of increasing orders, in that sense, derived from a succession of discoveries of new physical principle, each of which "brings us closer to the truth by one dimension" (in Gauss's words), is essential to Leibniz's calculus, and is even implicit in Leibniz's conception of the decimal system.

With these observations in mind, we can better appreciate some of the developments following Gauss's successful forecast of the orbit of Ceres.

On March 28, 1802, a short time after the rediscovery of Ceres by several astronomers in December 1801 and January 1802, precisely confirming Gauss's forecast, Gauss's friend Wilhelm Olbers discovered *another* small planet between Mars and Jupiter—the asteroid Pallas. Gauss immediately calculated the Pallas orbit from Olber's observations, and reported back with great excitement, that the two orbits, although lying in quite differ-

TABLE I. *The ratio of the squares of the periodic times of any two planets, is equal to the ratio of the cubes of the corresponding mean distances to the sun, which are equal to the semi-major axes of the orbits.*

Planet	Mean distance to sun A (in A.U.*)	Time T (in Earth yrs)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.862
Saturn	9.534	29.456

* 1 Astronomical Unit (A.U.) = 1 Earth-sun distance

ent planes, had nearly exactly the same periodic times, and appeared to cross each other in space! Gauss wrote to Olbers:

> In a few years, the conclusion [of our analysis of the orbits of Pallas and Ceres–JT] might either be, that Pallas and Ceres once occupied the same point in space, and thus doubtlessly formed parts of one and the same body; or else that they orbit the sun undisturbed, and with precisely equal periods . . . [in either case,] these are phenomena, which to our knowledge are unique in their type, and of which no one would have had the slightest dream, a year and a half ago. To judge by our human interests, we should probably not wish for the first alternative. What panic-stricken anxiety, what conflicts between piety and denial, between rejection and defense of Divine Providence, would we not witness, were the possibility to be supported by fact, that a planet can be annihilated? What would all those people say, who like to base their academic doctrines on the unshakable permanence of the planetary system, when they see, that they have built on nothing but sand, and that all things are subject to the *blind* and *arbitrary play* of the forces of Nature! For my part, I think we should refrain from such conclusions. I find it almost wanton arrogance, to take as a measure of eternal wisdom, the perfection or imperfection which we, with our limited powers and in our caterpillar-like stage of existence, observe or imagine to observe in the material world around us.

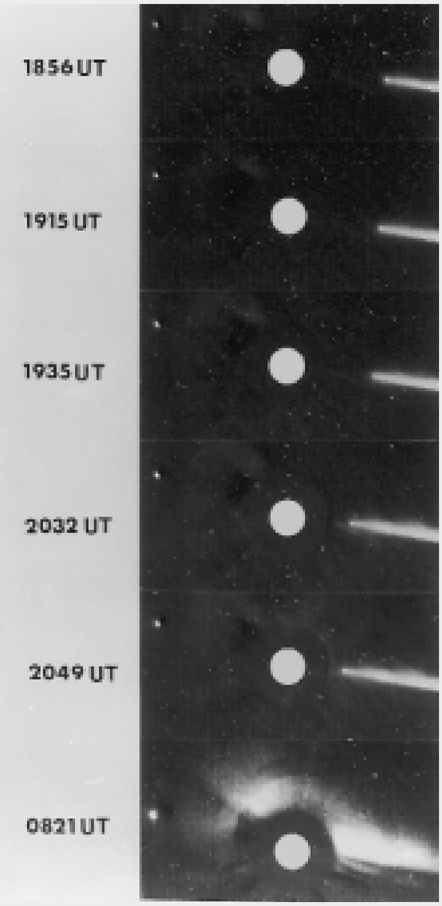


FIGURE 7.3.
Aug. 30, 1979: Comet Howard-Koomen-Michels streaks toward the sun at a speed of about 640,000 mph, trailed by a tail of dust and gas more than three million miles in length. A white disk has been added to the central area of the time-lapsed images, to show the size and location of the sun (which is masked by an opaque disk in the orbiting camera). The bright spot in the upper left corner of the photographs is the planet Venus. (Photo: U.S. Naval Research Laboratory)

The discovery of Ceres and Pallas, as probably the largest fragments of what had once been a larger planet, orbiting between Mars and Jupiter, helped dispose of the myth of "eternal tranquility" in the heavens. Indeed, we have good reason to believe, that cataclysmic events have occurred in the solar system in past, and might occur in the future. On an astrophysical scale, thanks to progress in the technology of astronomical observation, we are ever more frequent witnesses to a variety of large-scale events unfolding on short time scales. This includes the disappearance of entire stars in supernova explosions. The first well-documented case of this—the supernova which gave birth to the famous Crab Nebula—was recorded by Chinese astronomers in the year 1054. But, even within the boundaries of our solar system, dramatic events are by no means so exceptional as most people believe.

Apart from the hypothesized event of an explosion of a planet between Mars and Jupiter, made plausible by the discovery of the asteroid belt, collisions with comets and other interplanetary bodies are relatively frequent. We witnessed one such collision with Jupiter not long ago. Another example is the collision of the comet Howard-Koomen-Michels with the surface of the sun, which occurred around midnight on Aug. 30, 1979. This spectacular event was photographed by an orbiting solar observatory of the U.S. Naval Research Laboratory. **(Figure 7.3)** The comet's trajectory (which ended at the point of impact) was very nearly a perfect, parabolic Keplerian orbit, whose perihelion unfortunately was located closer to the center of the sun, than the sun's own photosphere surface! A century earlier, the Great Comet of 1882 was torn apart, as it passed within 500,000 kilometers of the photosphere, emerging as a cluster of five fragments.

Beyond these sorts of events, that appear more or less accidental and without great import for the solar system as a whole, it is quite conceivable, that even the present arrangement of the planetary orbits might undergo more or less dramatic and rapid changes, as the system passes from one Keplerian ordering to another.

—*JT*

CHAPTER 8

Parabolic and Hyperbolic Orbits

We have one last piece of business to dispose of, before we launch into Gauss's solution in Chapter 9. We have to devise a way of extending Kepler's constraints to the case of the parabolic and hyperbolic orbits, inhabited by comets and other peculiar entities in our solar system.

Comets and Non-Cyclical Orbits

During Kepler's time, the nature and motion of the comets was a subject of great debate. From attempts to measure the "daily parallax" in the apparent positions of the Great Comet of 1577, as observed at different times of the day (i.e., from different points of observation, as determined by the rotation and orbital motion of the Earth), the Danish astronomer Tycho Brahe had concluded that the distance from the Earth to the comet must be at least four times that of the distance between the Earth and Moon. Tycho's measurement was viciously attacked by Galileo, Chiaramonti, and others in Paolo Sarpi's Venetian circuits. Galileo *et al.* defended the generally accepted "exhalation theory" of Aristotle, according to which the comets were supposed to be phenomena generated inside the Earth's atmosphere. Kepler, in turn, refuted Galileo and Chiaramonti point-by-point in his late work, *Hyperaspistes,* published 1625. But Kepler never arrived at a satisfactory determination of comet trajectories.

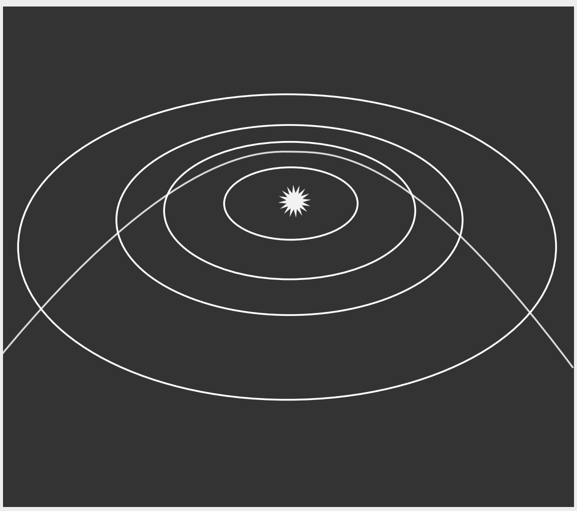


FIGURE 8.1. *The parabolic path of a comet, crossing the elliptical orbits of Mercury, Venus, Earth, and Mars.*

If Johann von Maedler's classic account is to be believed, the hypothesis of parabolic orbits for comets was first put forward by the Italian astronomer Giovanni Borelli in 1664, and later confirmed by the German pastor Samuel Doerfel, in 1681.

By the time of Gauss, it was definitively established that parabolic and even hyperbolic orbits were possible in our solar system, in addition to the elliptical orbits originally described by Kepler. In the introduction to his *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections,* Gauss emphasizes that the discovery of parabolic and hyperbolic orbits had added an important new dimension to astronomy. Unlike the periodic, cyclical motion of a planet in an elliptical orbit, a body moving in a parabolic or hyperbolic orbit traverses its trajectory only once.* This poses the problem of constructing the equivalent of Kepler's constraints for the case of non-elliptical orbits. **(Figure 8.1)**

The existence of parabolic and hyperbolic orbits, in fact, highlighted a paradox already implicit in Kepler's own derivation of his constraints, and to which Kepler himself pointed in the *New Astronomy.*†

In his initial formulation of what became known as the Second Law, Kepler spoke of the "time spent" at any given position of the orbit as being proportional to the "radial line" from the planet to the sun. He posed to future geometers the problem of how to "add up" the radial lines generated in the course of the motion, which seemed "infinite in number." Later, Kepler replaced the radial lines with the notion of sectoral areas described around the sun during the motion of "infinitely small" intervals of time. He prescribed that the ratios of those infinitesimal areas to the corresponding elapsed times, be the same for all parts of the orbit. Since that relationship is preserved during the entire process, during which such

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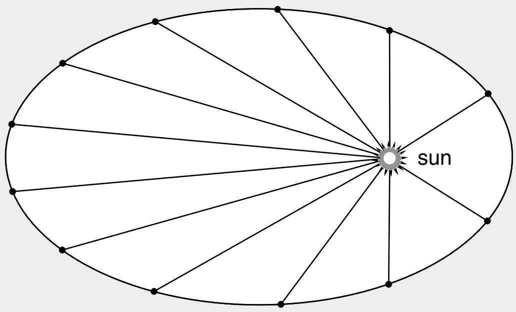
* A certain percentage of the comets have essentially elliptical orbits and relatively well-defined periods of recurrence. A famous example is Halley's Comet (which Halley apparently stole from Flamsteed), with a period of 76 years. Generally, however, the trajectories even of the recurring comets are unstable; they depend on the "conjunctural" situation in the solar system, and never exactly repeat. In the idealized case of a parabolic or hyperbolic orbit, the object never returns to the solar system. In reality, "parabolic" and "hyperbolic" comets sometimes return in new orbits.

† SEE extracts from Kepler's 1609 *New Astronomy*, pp. 24-25.

FIGURE 8.2. *Kepler's "area law." The ratios of elapsed times are proportional to the ratios of swept-out areas.*



"infinitesimal" areas accumulate to form a macroscopic area in the course of continued motion, it will be valid for any elapsed times whatever.

The result is Kepler's final formulation of the Second Law, which very precisely accounts for the manner in which the rate of angular displacement of a planet around the sun actually slows down or speeds up in the course of an orbit. **(Figure 8.2)**

However, while specifying, in effect, that the *ratios* of elapsed times are proportional to the ratios of swept-out areas, the Second Law says nothing about their absolute magnitudes. The latter depend on the dimensions of the orbit as a whole, a relationship manifested in the progressive, stepwise decrease in the overall periods and average velocities of the planets, as we move outward away from the sun, i.e., from Mercury, to Venus, the Earth, Mars, Jupiter, and so on. (SEE Figure 7.1b) In his *Harmony of the World* of 1619, Kepler characterized that overall relationship by what became known as the Third Law, demonstrating that the squares of the periodic times are proportional to the cubes of the semi-major axes of the orbits or, equivalently: The periodic times are proportional to the three-halves powers of the semi-major axes (SEE Table I, page 36).

Thus, the Third Law addresses the integrated result of an entire periodic motion, while the Second Law addresses the changes in rate of motion subsumed within that cycle. The relationship of the two, in terms of Kepler's original approach, is that of an integral to a differential.

What happens to the Third Law in the case of a parabolic or hyperbolic orbit? In such case, the motion is no longer periodic, and the axis of the orbit has no assignable length. The periodic time and semi-major axes have, in a sense, both become "infinite." On the other hand, the motion of comets must somehow be coherent with the Keplerian motion of the main planets, just as there exists an overall coherence among all ellipses, parabolas, and hyperbolas, as subspecies of the family of conic sections. In fact, the motions of the comets are found to follow Kepler's Second Law to a very high degree of precision. That suggests a very simple consideration: How might we characterize the relationship between elapsed times and areas swept out, in terms of absolute values (and not only ratios), *without* reference to the length of a completed period? We can do that quite easily, thanks to Kepler's own work, by combining all three of Kepler's constraints.

Gauss's Constraints

Kepler's Second Law defines the ratio of the area swept out around the sun, to the elapsed time, as an unchanging, characteristic value for any given orbit. For an elliptical orbit, Kepler's Third Law allows us to determine the value of that ratio, by considering the special case of a single, completed orbital period. The area swept out during a complete period, is the entire area of the ellipse, which (as was already known to Greek geometers) is equal to $\pi \times A \times B$, where A and B are the semi-major and semi-minor axes, respectively. The elapsed time is the duration T of a whole period, known from Kepler's Third Law to be equal to $A^{3/2}$, when the semi-major axis and periodic time of the Earth's orbit are taken as units. The quotient of the two is $\pi \times A \times (B/A^{3/2})$, or in other words $\pi \times (B/\sqrt{A})$. Now, the quotient $B/\sqrt{A}$ has a special significance in the geometry of elliptical orbits [SEE **Appendix** (l)]: Its square, B^2/A, is equal to the "half-parameter" of the ellipse, which is half the width of the ellipse as measured across the focus in the direction perpendicular to the major axis. **(Figure 8.3a)** The importance of the half-parameter, which is equivalent to the radius in the case of a circle, lies in the fact that it has a definite meaning not only for circular and elliptical orbits, but also for parabolic and hyperbolic ones. **(Figures 8.3b and c)** The "orbital parameter" and "half-parameter" played an important role in Gauss's astronomical theories.

We can summarize the result just obtained as follows: For elliptical orbits, at least, the value of Kepler's ratio of area swept out to elapsed time—a ratio which is constant for any given orbit—comes out to be

$$\pi \times \sqrt{H},$$

where H is the half-parameter of the orbit. Unlike a periodic time and finite semi-major axis, which exist for elliptical but not parabolic or hyperbolic orbits, the "parameter" does exist for all three. Does the corresponding relationship actually hold true, for the actually observed trajectories of comets? It does, as was verified, to a high degree of accuracy, by Olbers and earlier

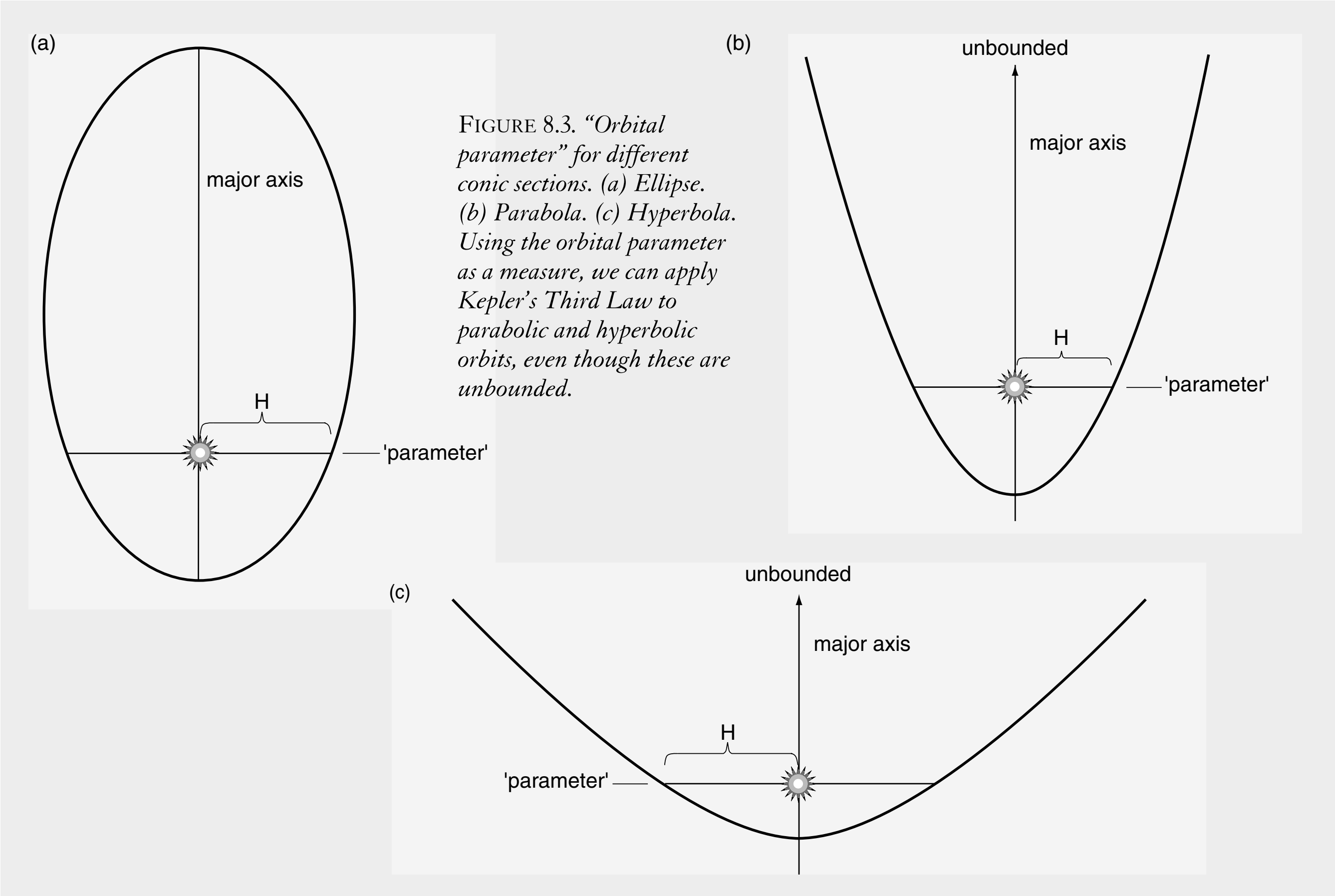


FIGURE 8.3. *"Orbital parameter" for different conic sections. (a) Ellipse. (b) Parabola. (c) Hyperbola. Using the orbital parameter as a measure, we can apply Kepler's Third Law to parabolic and hyperbolic orbits, even though these are unbounded.*

astronomers prior to Gauss's work.

The purpose of this exercise was to provide a replacement for Kepler's Third Law, which applies to parabolic and hyperbolic orbits, as well as to elliptical ones. We have succeeded. The constant of proportionality, connecting the ratio of area and time on the one side, and the square root of the "parameter" on the other, came to be known as "Gauss's constant." Taking the orbit of Earth as a unit, the constant is equal to π.

With one slight, additional modification, whose details need not concern us here,* the following is the generalized form of Kepler's constraints, which Gauss sets forth at the outset of his *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections.*

Gauss emphasizes that they constitute "the basis for all the investigations in this work":

(i) The motion of any given celestial body always occurs in a constant plane, upon which lies, at the same time, the center of the sun.

(ii) The curve described by the moving body is a conic section whose focus lies at the center of the sun.

(iii) The motion in that curve occurs in such a way, that the sectoral areas, described around the sun during various time intervals, are proportional to those time intervals. Thus, if one expresses the times and areas by numbers, the area of any sector, when divided by the time during which that sector was generated, yields an unchanging quotient.

(iv) For the various bodies orbiting around the sun, the corresponding quotients are proportional to the square roots of the half-parameters of the orbits.

—JT

* In his formulation in the *Theory of the Motion,* Gauss includes a factor correcting for a slight effect connected with the "mass ratio" of the planet to the sun. That effect, manifested in a slight increase in Kepler's ratio of area to elapsed time, becomes distinctly noticeable only for the larger planets, especially Jupiter, Saturn, and Uranus. The "mass," entering here, does not imply Newton's idea of some self-evident quality inhering in an isolated body. Rather, "mass" should be considered as a complex physical effect, measurable in terms of slight discrepancies in the orbits, i.e., as an additional dimension of curvature involving the relationship of the orbit, as singularity, to the entire solar system.

CHAPTER 9

Gauss's Order of Battle

Now, let us join Gauss, as he thinks over the problem of how to calculate the orbit of Ceres. Gauss had at his disposal about twenty observations, made by Piazzi during the period from Jan. 1 to Feb. 11, 1801. The data from each observation consisted of the specification of a moment in time, precise to a fraction of a second, together with two angles defining the precise direction in which the object was seen at that moment, relative to an astronomical system of reference defined by the celestial sphere, or "sphere of the fixed stars." Piazzi gave those angles in degrees, minutes, seconds, and tenths of seconds of arc.

In principle, each observation defined a line through space, starting from the location of Piazzi's telescope in space at the moment of the observation—the latter determinable in terms of the Earth's known rotation and motion around the sun—and directed along the direction defined by Piazzi's pair of angles. Naturally, Gauss had to make corrections for various effects such as the precession of the Earth's axis, aberration and refraction in the Earth's atmosphere, and take account of possible margins of error in Piazzi's observations.

Although the technical execution of Gauss's solution is rather involved, and required a hundred or more hours of calculation, even for a master of analysis and numerical computation such as Gauss, the basic method and principles of the solution are in principle quite elementary. Gauss's tactic was, first, to determine a relatively rough approximation to the unknown orbit, and then to progressively refine it, up to a high degree of precision.

Gauss's procedure was based on using only three observations, selected from Piazzi's data. Gauss's original choice consisted of the observations from Jan. 2, Jan. 22, and Feb. 11. (Figure 1.1) Later, Gauss made a second, definitive round of calculations, based on using the observations of Jan. 1 and Jan. 21, instead of Jan. 2 and Jan. 22.

Overall, Piazzi's observations showed an apparent retrograde motion from the time of the first observation on Jan. 1, to Jan. 11, around which time Ceres reversed to a forward motion. Most remarkable, was the size of the angle separating Ceres' apparent direction from the plane of the ecliptic—an angle which grew from about 15° on Jan. 1, to over 18° at the time of Piazzi's last observation. That wide angle of separation from the ecliptic, together with the circumstance, that all the major planets were known to move in planes much closer to the ecliptic, prompted Piazzi's early suspicion that the object might turn out to be a comet.

Gauss's first goal, and the most challenging one, was *to determine the distance of Ceres from the Earth,* for at least one of the three observations. In fact, Gauss chose the *second* of the unknown distances—the one corresponding to the intermediate of the three selected observations—as the prime target of his efforts. Finding that distance essentially "breaks the back" of the problem. Having accomplished that, Gauss would be in a position to successively "mop up" the rest.

In fact, Gauss used his calculation of that value to determine the distances for the first and third observations; from that, in turn, he determined the corresponding spatial positions of Ceres, and from the two spatial conditions and the corresponding time, he calculated a first approximation of the orbital elements. Using the coherence provided by that approximate orbital calculation, he could revise the initial calculation of the distances, and obtain a second, more precise orbit, and so on, until all values in the calculation became coherent with each other and the three selected observations.

The discussion in Chapter 2 should have afforded the reader some appreciation of the enormous ambiguity contained in Piazzi's observations, when taken at face value. Piazzi saw only a faint point of light, only a "line of sight" *direction,* and nothing in any of the observations *per se,* permitted any conclusion whatsoever about how far away the object might be. It is only by analyzing the *intervals* defined by all three observations taken together, on the basis of the underlying, Keplerian curvature of the solar system, that Gauss was able to reconstruct the reality behind what Piazzi had seen.

Polyphonic Cross-Voices

Gauss's opening attack is a masterful application of the kinds of synthetic-geometrical methods, pioneered by Gérard Desargues *et al.,* which had formed the basis of the revolutionary accomplishments of the Ecole Polytechnique under Gaspard Monge.

Firstly, of course, we must have confidence in the powers of Reason, that the Universe is composed in such a way, that the problem can be solved. Secondly, we must consider everything that might be relevant to the problem. We are not permitted to arbitrarily "simplify" the problem. We cannot say, "I refuse to consider this, I refuse to consider that."

To begin with, it is necessary to muster not only the relevant data, but above all the *complex of interrelationships*—potential polyphonic cross-voices!—underlying the three observations in relation to each other and (chiefly) the sun, the positions and known orbital motion of the Earth, the unknown motion of Ceres, and the "background" of the rest of the solar system and the stars.

Accordingly, denote the times of the three observations by t_1, t_2, t_3, the corresponding (unknown!) true spatial positions of Ceres by P_1, P_2, P_3, and the corresponding positions of the Earth (or more precisely, of Piazzi's observatory) at each of the three moments of observation, by E_1, E_2, E_3. Denote the position of the center of the sun by O. **(Figure 9.1)** We must consider the following relationships in particular:

(i) The three "lines of sight" corresponding to Piazzi's observations. These are the lines passing from E_1 through P_1, from E_2 through P_2, and from E_3 through P_3. As already noted, the observations tell us only the *directions* of those lines and, from knowledge of the Earth's motion, their points of origin, E_1, E_2 and E_3; but *not* the *distances* E_1P_1, E_2P_2 and E_3P_3.

(ii) The *elapsed times* between the observations, taken pairwise, i.e., $t_2–t_1$, $t_3–t_2$, and $t_3–t_1$, as well as the ratios or intervals of those elapsed times, for example $t_3–t_2 : t_2–t_1$, $t_2–t_1 : t_3–t_1$, $t_3–t_2 : t_3–t_1$, and the various permutations and inversions of these.

FIGURE 9.1. *Positions of the sun (O), Earth (E), and Ceres (P), at the three times of observation. P_1,P_2,P_3 must lie on lines of sight L_1,L_2,L_3, but their distances from Earth are not known.*

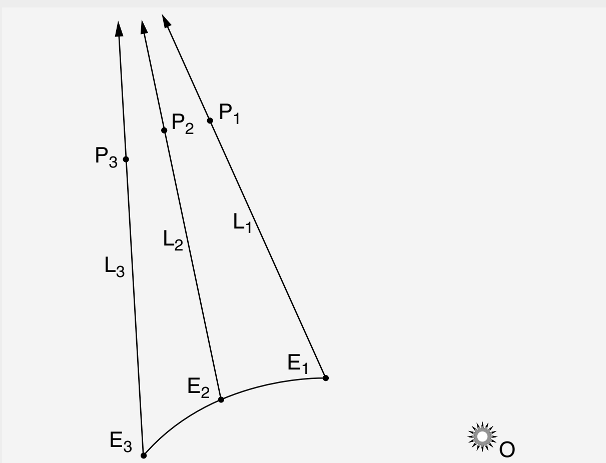
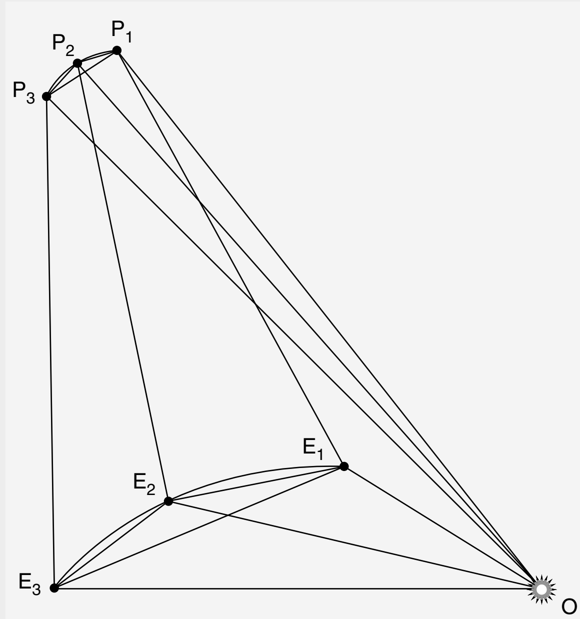


FIGURE 9.2. *Triangular relationships among Earth, Ceres, and the sun.*



(iii) The *orbital sectors* for the Earth and Ceres, corresponding to the elapsed times just enumerated, in relation to one another and the elapsed times.

(iv) The *triangles* formed by the positions of the Earth, Ceres and the sun, in particular the triangles OE_1E_2, OE_2E_3, OE_1E_3, and triangles OP_1P_2, OP_2P_3, OP_1P_3, representing relationships among the three positions of the Earth and of Ceres, respectively; plus the three triangles formed by the positions of the sun, the Earth and Ceres at each of the three times, taken together: OE_1P_1, OE_2P_2, OE_3P_3. Also, each of the line segments forming the sides of those triangles. **(Figure 9.2)**

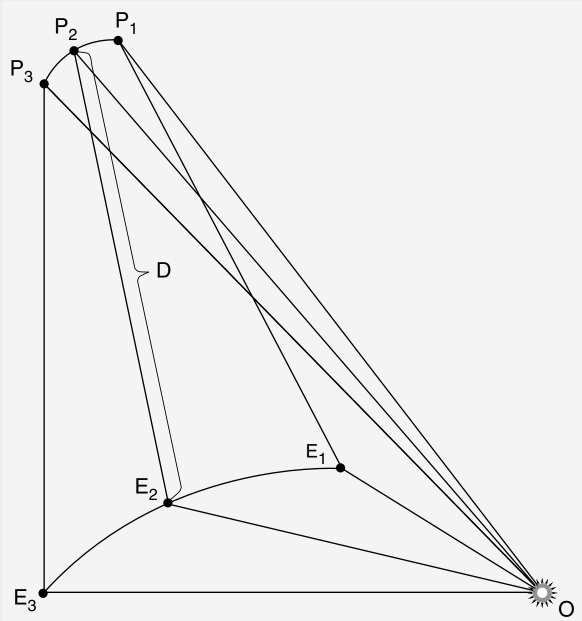
Each line segment must be considered, not as a noun but as a verb, a geometrical interval. For example, the segment E_1P$_1$ implies a potential action of displacement from E_1 to P_1. Displacement E_1P_1 is therefore not the same as P_1E_1.

(v) The relationships (including relationships of area) between the triangles OE_1E_2, OE_2E_3, OE_1E_3, as well as OP_1P_2, OP_2P_3, OP_1P_3 and the corresponding orbital sectors, as well as the elapsed times, in view of the Kepler/Gauss constraints.

Gauss's immediate goal, is to determine the second of those distances, the distance from E_2 to P_2. Call that critical unknown, "*D*." **(Figure 9.3)**

Although we shall not require it explicitly here, for his detailed calculations, Gauss, in a typical fashion, intro-

FIGURE 9.3. *Gauss's immediate goal: determine distance D from* E_2 *to* P_2.



duces a spherical projection into the construction, transferring the directions of all the various lines in the problem for reference to a single center. **(Figure 9.4)** Thereby, Gauss generates a new set of relationships, as indicated in Figure 9.4.

Faced with this bristling array of relationships, some readers might already be inclined to call off the war, before it has even started. Don't be a coward! Don't be squeamish! Nothing much has happened yet. However bewildering this complex of spatial relationships might appear at first sight, remember: everything is bounded by the curvature of what Jacob Steiner called "the organism of space." All relationships are generated by one and the same Gaussian-Keplerian principle of change, as embodied in the combination of motions of the Earth and Ceres, in particular. The apparent complexity just conceals the fact that we are seeing one and the same "One," reflected and repeated in many predicates.

As for the construction, it is all in our heads. Seen from the standpoint of Desargues, the straight lines are nothing but artifacts subsumed under the "polyphonic" relationships of the angles formed by the various directions, seen as "monads," located at the sun, Earth, and Ceres.

Somewhere within these relationships, the desired distance "*D*" is lurking. How shall we smoke it out? Might the answer not lie in looking for the footprints of a *differential of curvature* between the Earth's motion and the (unknown) motion of Ceres?

We shall discover Gauss's wonderfully simple solution in the following chapter.

—*JT*

FIGURE 9.4. *Gauss's spherical mapping. The directions of the lines* L_1, L_2, L_3 *are transferred to an imaginary sphere S, by drawing unit segments* l_1, l_2, l_3, *parallel to* L_1, L_2, L_3, *respectively, from the center c. In addition (although not shown here, for the sake of simplicity), Gauss transferred all the other directions in the problem—i.e., the directions of the lines* OE_1, OE_2, OE_3, *and* OP_1, OP_2, OP_3*—to the "reference sphere," thus obtaining a summary of all the angular relationships.*