CHAPTER 10

Closing In on Our Target

Gauss is a mathematician of fanatical determination, he does not yield even a hand's width of terrain. He has fought well and bravely, and taken the battlefield completely.

—Comment by Georg Friedrich von Tempelhoff, 1799.
Prussian General and Chief of Artillery, Tempelhoff was also known for his work in mathematics and military history. The youthful Gauss, who regarded him as one of the best German mathematicians, had sent him a pre-publication copy of his *Disquisitiones Arithmeticae.*

Although Gauss knew analytical calculation perhaps better than any other living person, he was sharply opposed to any mechanical use of it, and sought to reduce its use to a minimum, as far as circumstances allowed. He often told us, that he never took a pencil into his hand to calculate, before the problem had been completely solved by him in his head; the calculation appeared to him merely as a means by which to carry out his work to its conclusion. In discussions about these things, he once remarked, that many of the most famous mathematicians, including very often Euler, and even sometimes Lagrange, trusted too much to calculation alone, and could not at all times account for what they were doing in their investigations. Whereas he, Gauss, could affirm, that at every step he always had the goal and purpose of his operations precisely in mind, and never strayed from the path.

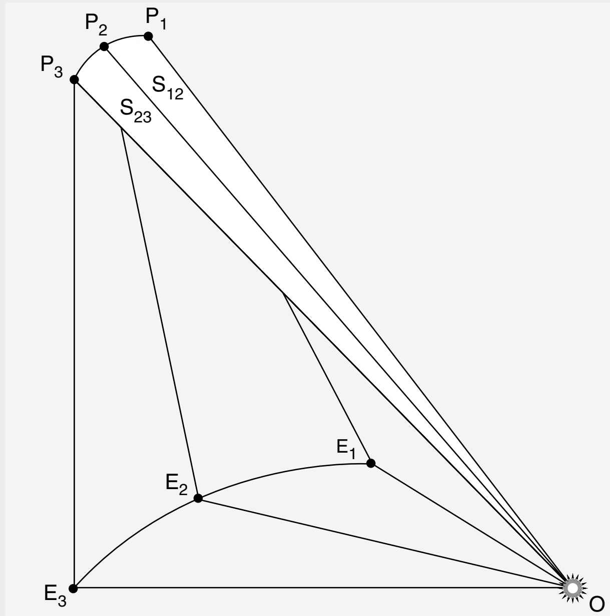
—Walther Sartorius von Waltershausen,
godson of Goethe and a student and close friend of Gauss, in a biographical sketch written soon after Gauss's death in December 1855.

In the last chapter, we mustered the key elements which must be taken into account to determine the Earth-Ceres distances and, eventually, the orbit of Ceres, from a selection of three observations, each giving a time and the angular coordinates of the apparent position of Ceres in the heavens at the corresponding instants.

Our suggested approach is to "read" the space-time intervals among the three chosen observations, as implicitly expressing a relationship between the curvatures of the orbits of Earth and Ceres. Then, compare the adduced differential, with the "projected" appearance, to derive the distances and the positions of the object.

To carry out this idea, Gauss first focusses on the manner in which the second ("middle") position of each planet is related to its first and third (i.e., "outer") positions. In other words: How is P_2 related to P_1 and P_3? And, what is the distinction of the relation of P_2 to P_1 and P_3, in comparison with that of E_2 to E_1 and E_3? **(Figure 10.1)**

FIGURE 10.1. *Sectoral areas S_{12} and S_{23}, swept out as Ceres moves, respectively, from P_1 to P_2, and from P_2 to P_3.*



Thanks to our knowledge of the overall curvature of the solar system, embodied *in part* in the Gauss-Kepler constraints, we can say something about those questions, even before knowing the details of Ceres' orbit.

To wit: Regard P_2 and E_2 as singularities resulting from *division of the total action of the solar system,* which carries Ceres from P_1 to P_3, and simultaneously carries the Earth from E_1 to E_3, during the time interval from t_1 to t_3. In both cases, the Gauss-Kepler constraints tell us, that the *sectoral areas* swept out by the two motions, are proportional to the elapsed times. The latter, in turn, are known to us, from Piazzi's observations.

Explore this matter further, as follows. Concentrating first on Ceres, write, as a shorthand:

S_{12} = area of orbital sector swept out by Ceres from P_1 to P_2,

S_{23} = area of orbital sector from P_2 to P_3,

S_{13} = area of orbital sector from P_1 to P_3.

According to the Gauss-Kepler constraints, the ratios

$S_{12} : t_2–t_1$, $S_{23} : t_3–t_2$, and $S_{13} : t_3–t_1$,

which are equivalent to the fractional expressions more easily used in computation

$$\frac{S_{12}}{t_2-t_1}, \quad \frac{S_{23}}{t_3-t_2}, \quad \text{and} \quad \frac{S_{13}}{t_3-t_1},$$

all have the same identical value, namely, the product of Gauss's constant (in our context equal to π) and the square root of Ceres' orbital parameter. (SEE Chapter 8) The analogous relationships obtain for the Earth. Now, we don't know the value of Ceres' orbital parameter, of course; nevertheless, the above-mentioned proportionalities are enough to determine key "cross"-ratios of the areas and times among themselves, without reference to the orbital parameter. For example, the just-mentioned circumstance that

S_{12} : elapsed time $t_2–t_1$: : S_{23} : elapsed time $t_3–t_2$

(the " : : " symbol means an equivalence between two ratios), has as a consequence, that the ratio of those areas must be equal to the ratio of the elapsed times, or in other words:

$$\frac{S_{12}}{S_{23}} = \frac{t_2-t_1}{t_3-t_2},$$

and similarly for the various permutations of orbital positions 1, 2, and 3.

Now, we can compute the elapsed times, and their ratios, from the data supplied by Piazzi, for the observations chosen by Gauss. The specific values are not essential to the general method, of course, but for concreteness, let's introduce them now. In terms of "mean solar time," the times given by Piazzi for the three chosen observations, were as follows:

t_1 = 8 hours 39 minutes and 4.6 seconds p.m. on Jan. 2, 1801.
t_2 = 7 hours 20 minutes and 21.7 seconds p.m. on Jan. 22, 1801.
t_3 = 6 hours 11 minutes and 58.2 seconds p.m. on Feb. 11, 1801.

The circumstance, that t_2 is nearly half-way between t_1 and t_3, yields a certain advantage in Gauss's calculations, and is one of the reasons for his choice of observations. Calculated from the above, the elapsed times are:

$t_2–t_1$ = 454.68808 hours,

$t_3–t_2$ = 478.86014 hours, and

$t_3–t_1$ (the sum of the first two) = 933.54842 hours.

Calculating the various ratios, and taking into account what we just observed concerning the implications of the Gauss-Kepler constraints, we get the following conclusion:

$$\frac{S_{12}}{S_{23}} = \frac{t_2-t_1}{t_3-t_2} = 0.94952,$$

$$\frac{S_{12}}{S_{13}} = \frac{t_2-t_1}{t_3-t_1} = 0.48705,$$

$$\frac{S_{23}}{S_{13}} = \frac{t_3-t_2}{t_3-t_1} = 0.51295.$$

Everything we have said so far, including the numerical values just derived, applies just as well to the Earth, as to Ceres. We merely have to substitute the areas swept out by the Earth in the corresponding times. Of course, in the case of the Earth, we know its positions and orbital motion quite precisely; here, the ratios of the sectoral areas tell us nothing essentially new. For Ceres, whose orbit is *unknown* to us, our application of the "area law" has placed us in a paradoxical situation: Without, for the moment, having any way to calculate the orbit and the areas of the orbital sectors themselves, we now have precise values for the *ratios* of those areas!

How could we use those ratios, to derive the orbit of Ceres? Not in any linear way, obviously, because the same values apply to the Earth and *any* planet moving according to the Gauss-Kepler constraints. The key, here, is not to think in terms of "getting to the answer" by some "straight-line" procedure. Rather, we have to think of progressing in *dimensionalities,* just as in a battle we strive to increase the freedom of action of our own forces, while progressively reducing that of the enemy forces. So, at each stage of our determination of the Ceres orbit, we try to increase what we know by one or more dimensions, while reducing the indeterminacy of what we must know, but don't yet know, to a corresponding extent. We don't have to worry about how the orbital values will finally be calculated, in the end. It is enough to know, that by proceeding in the indicated way, the values will eventually be "pinned down" as a matter of course.

So, our acquiring the values for the *ratios* of the sectoral areas generated by Ceres' motion, does not in itself lead to the desired orbital determination; but, in the context of the whole complex of relationships, we have closed in on our target by at least one "dimensionality."

Accordingly, return once more to the relationship of the intermediate position of Ceres (P_2), to the outside positions (P_1 and P_3). Introduce a new tactic, as follows.

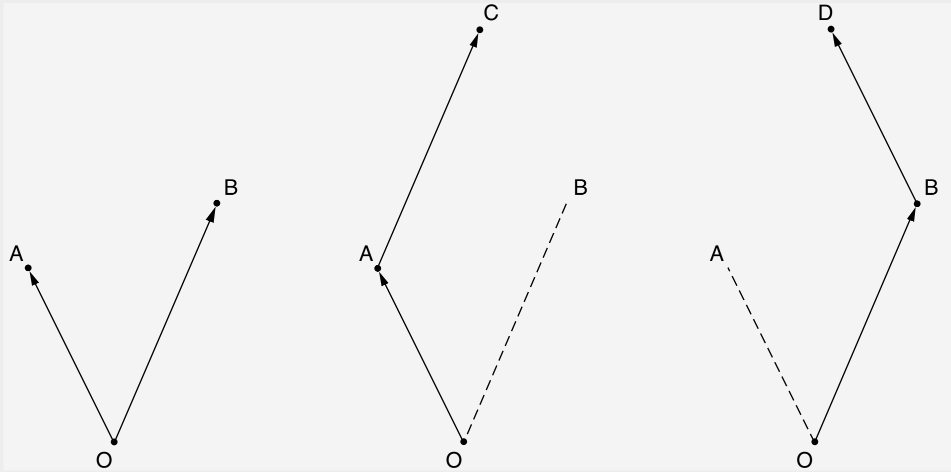


FIGURE 10.2. *The famous "parallelogram law" for combination of displacements OA and OB, assumes that the result of the combination does not depend on the order in which the displacements are carried out—i.e., that C and D coincide. Gauss considered that this might only be approximately true, and that the parallelogram law might break down when the displacements are very large.*

The Harmonic Ordering of Action in Space

Among most elementary characteristics of the "organism of space," is the manner in which the result of a series of displacements, is related to the individual displacements making up that series. This concerns us very much in the case in point. For example, the *apparent* position of Ceres, as seen from the Earth at any given moment, corresponds to the direction, in space, of the line segment from the Earth to Ceres. The latter, seen as a geometrical interval or displacement, can be represented as a differential between two other spatial intervals or displacements, namely the interval from the sun to the Earth, "subtracted," in a sense, from the interval from the sun to Ceres. Or, to put it another way: the displacement from the sun to Ceres, can be broken down as the resultant or sum of the displacement from the sun to the Earth, following by the displacement from the Earth to Ceres. Similarly, we have to take account of successive displacements corresponding to the motions P_1 to P_2, P_2 to P_3, E_1 to E_2, E_2 to E_3, etc.

Now, this *apparently* self-evident mode of combining displacements, involves an *implicit assumption,* which Gauss was well aware of. If I have two displacements from a *common locus,* say from the O (i.e., the center of the sun) to a location A, and from the O to location B, then I might envisage the combination or addition of the displacements in either of the following two ways **(Figure 10.2)**: I might apply the first displacement, to go from O to A, and then go from A to a third location, C, by displacing *parallel* to the second displacement from

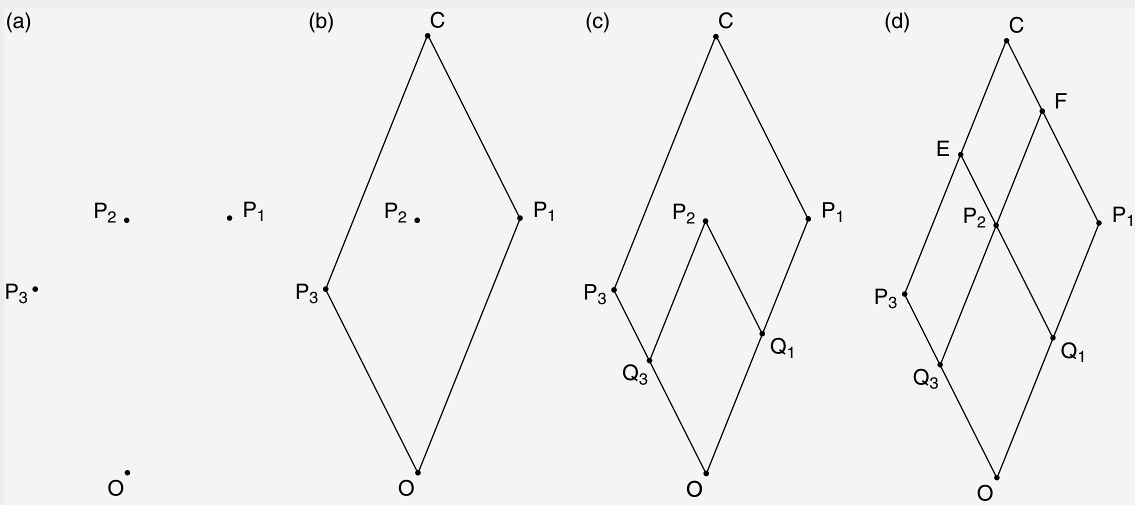


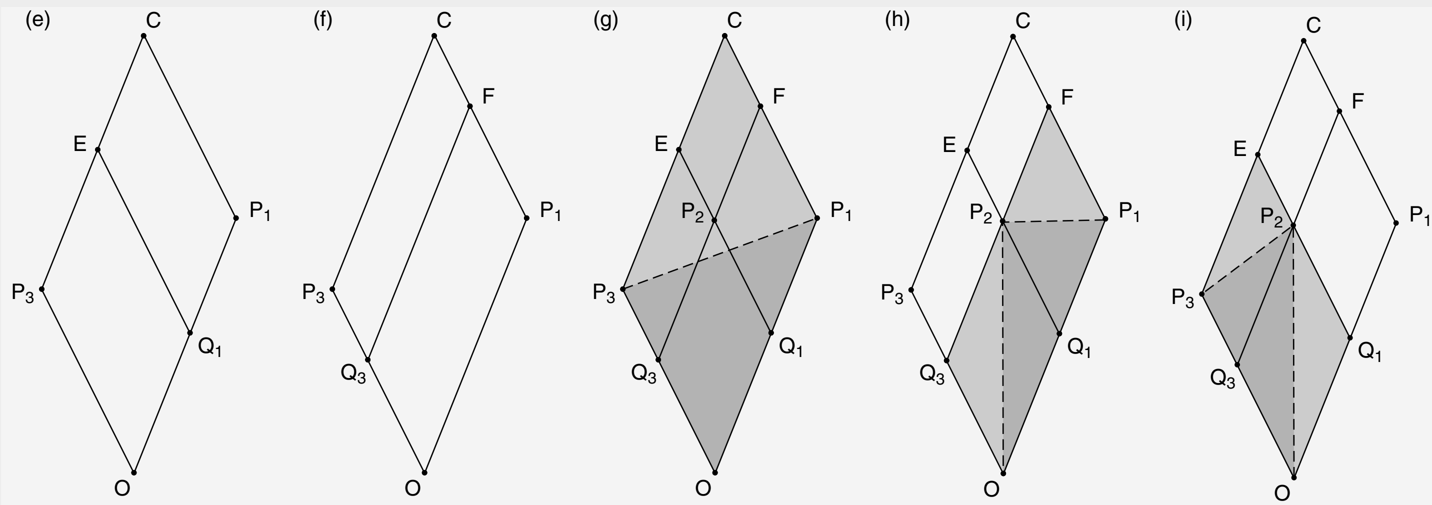
FIGURE 10.3. *Derivation of the location of P_2 by parallel displacements along directions OP_1 and OP_3.*

the O to B, and by the same distance. The displacement from A to C is parallel and congruent to that from O to B, and can be considered as equivalent to the latter in that sense. Or, I might operate the displacements in the *opposite order;* moving first from O to B, and *then* moving parallel and congruent with OA, from B to a point D. The obvious assumption here is, that the two procedures produce the same end result, or in other words, that C and D will be the same location. In that case, the displacements OA, AC, OB, BC will form a *parallelogram* whose opposing pairs of sides are congruent and parallel line segments.

Could it happen, that C and D might actually turn out to be different, in reality? Gauss himself sought to define large-scale experiments using beams of light, which might produce an anomaly of a similar sort. Gauss was convinced, that Euclidean geometry is nothing but a useful approximation, and that the actual characteristics of visual space, are derived from a higher, "anti-Euclidean" curvature of space-time. Such an "anti-Euclidean" geometry, is already implied by the Keplerian harmonic ordering of the solar system, and would be demonstrated, again, by Wilhelm Weber's work on electromagnetic singularities in the microscopic domain, as well as the work of Fresnel on the nonlinear behavior of light "in the small." Hence, once more, the irony of Gauss's applying elementary constructions of Euclidean geometry, to the orbital determination of Ceres. Gauss's use of such constructions, is informed by the primacy of the "anti-Euclidean" geometry, in which his mind is already operating.

Turning to the relationship of P_2 to P_1 and P_3, the question naturally arises: Is it possible to describe the location P_2, as the combined result of a pair of displacements, along the directions of OP_1 and OP_3, respectively? **(Figure 10.3)** The possibility of such a representation, is already implicit in the fact, emphasized by Gauss in his reformulation of Kepler's constraints, that the orbit of any planet lies in a *plane* passing through the center of the sun. A plane, on the other hand, is a simplified representation of a "doubly extended manifold," where all characteristic modes of displacement are reducible to two principles or "dimensionalities." On the elementary geometrical level, this means, that out of any *three* displacements, such as OP_1, OP_2, and OP_3, one must be reducible to a combination of the other two, or at least of displacements along the directions defined by the other two. In fact, it is easy to construct such a decomposition, as follows.

Start with only the two displacements OP_1 and OP_3. Combine the two displacements, in the manner indicated above, to generate a point C, as the fourth vertex of a parallelogram consisting of OP_1, OP_3, P_1C, and P_3C. **(Figure 10.3b)** Now, apart from extreme cases (which we need not consider for the moment), the position P_2 will lie *inside* the parallelogram. We need only "project" P_2 onto each of the "axes" OP_1, OP_3 by lines parallel with the other axis. **(Figure 10.3c)** In other words, draw a parallel to OP_1 through P_2, intersecting the segment OP_3 at a point Q_3, and intersecting the parallel segment P_1C at a point F. Draw a parallel to OP_3 through P_2, intersecting the segment OP_1 at a point Q_1 and the parallel segment P_3C at a point E. The result of this construction, is to create several sub-parallelograms, including one with sides OQ_1, Q_1P_2, OQ_3, Q_3P_2, and having P_2 as a vertex. **(Figure 10.3d)**



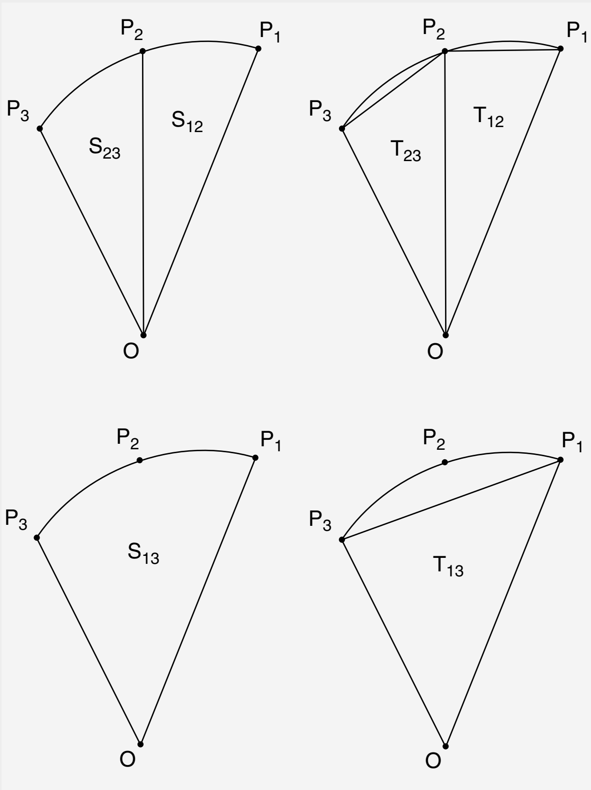


FIGURE 10.4. *Orbital sectors S_{12},S_{23},S_{13}, and their corresponding triangular areas T_{12}, T_{23}, T_{13}.*

Examining this result, we see that the displacement OP_2, which corresponds to the diagonal of the above mentioned sub-parallelogram, is equivalent, *by construction,* to the combination or sum of the displacements OQ_1 and OQ_3, the latter lying along the axes defined by P_1 and P_3. We have thus expressed the position of P_2 in terms of P_1, P_3, and the two other division points Q_1 and Q_3.

This suggests a new question: Given, that all these constructions are hypothetical in character, since the positions of P_1, P_2, and P_3 are yet unknown to us, do Piazzi's observations together with the Gauss-Kepler constraints, allow us to draw any conclusions of interest, concerning the location of the points Q_1 and Q_3, or at least the shape and *proportions* of the sub-parallelogram $OQ_1P_2Q_3$, in relation to the parallelogram OP_1CP_3?

Aha! Why not have a look at the relationships of *areas* involved here, which must be related in some way to the areas swept out during the orbital motions. First, note that the line Q_1E, which was constructed as the parallel to OP_3 through P_2, divides the area of the whole parallelogram OP_1CP_3 according to a specific proportion, namely that defined by the ratio of the segment OQ_1, to the larger segment OP_1. **(Figure 10.3e)** Similarly, the line Q_3F divides the area of the whole parallelogram according to the proportion of OQ_3 to OP_3. **(Figure 10.3f)** Or, conversely: the ratios $OQ_1:OP_1$ and $OQ_3:OP_3$ are the same, respectively, as the *ratios of the areas* of the sub-parallelograms OQ_1EP_3 and OQ_3FP_1, to the whole parallelogram OP_1CP_3.

What are those areas? Examining the triangles generated by our division of the parallelogram, and by the segments P_1P_2, P_2P_3, and P_1P_3, observe the following: The triangle OP_1P_3 makes up exactly *half* the area of the whole parallelogram OP_1CP_3. **(Figure 10.3g)** The triangle OP_1P_2 makes up half the area of the sub-parallelogram OQ_3FP_1 **(Figure 10.3h)**, and the triangle OP_2P_3 makes up exactly half the area of the parallelogram OQ_1EP_3. **(Figure 10.3i)** Consequently, the ratios of the parallelogram areas, which in turn are the ratios by which Q_3 and Q_1 divide the segments OP_3 and OP_1, respectively, are nothing other than the ratios of the triangular areas OP_1P_2 and OP_2P_3, respectively, to the triangular area OP_1P_3. As a shorthand, denote those areas by T_{12}, T_{23}, and T_{13}, respectively. **(Figure 10.4)**

This brings us to a critical juncture in Gauss's whole solution: How are the areas of the triangles, just mentioned, related to the corresponding sectors, swept out by the motion of Ceres, and whose ratios are known to us?

Comparing T_{12} with S_{12}, for example, we see that the difference lies only in the relatively small area, enclosed between the orbital arc from P_1 to P_2, and the line segment connecting P_1 and P_2. The magnitude of that area, is an effect of the curvature of the orbital arc. Now, if we knew what that was, we could calculate the ratios of the triangular areas from the known ratios of the sectors; and from that, we would be in possession of the ratios defining the division of OP_1 and OP_3 by the points Q_1 and Q_3. Those ratios, in turn, express the spatial relationship between the intermediate position P_2 and the outer positions P_1 and P_3. As we shall see in Chapter 11, that would bring us very close to being able to calculate the distance of Ceres from the Earth, by comparing such an adduced spatial relationship, to the observed positions as seen from the Earth.

Fine and good. But, what do we know about the curvature of the orbital arc from P_1 to P_3? Was it not exactly the problem we wanted to solve, to determine what Ceres' orbit is? Or, do we know something more about the curvature, even without knowing the details of the orbit?

—JT

Chapter 11

Approaching the *Punctum Saliens*

We are nearing the *punctum saliens* of Gauss's solution. The constructions in this and the following chapters are completely elementary, but highly polyphonic in character.

Let us briefly review where we stand, and add some new ideas in the process.

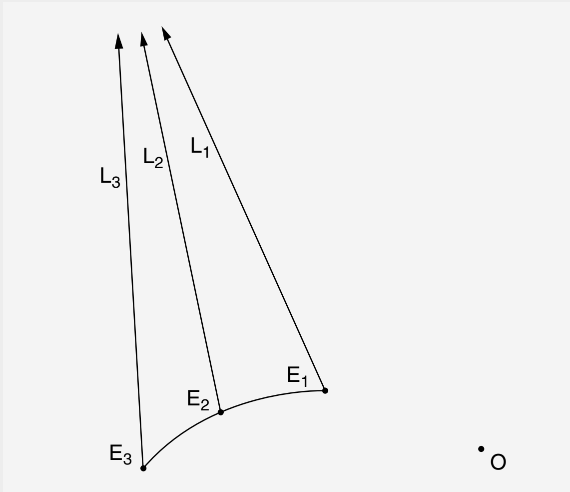
Recall the nature of the problem: We have three observations by Piazzi, reporting the apparent position of Ceres in the sky, as seen from the Earth, at three specified moments of time, approximately twenty days apart. The first task set by Gauss, is to determine the distance of Ceres from the Earth for at least one of those observations.

Two "awesome" difficulties seemed to stand in our way:

First, the observations of the motion of Ceres, were made from a point which is not fixed in space, but is also moving. The position and apparent motion of Ceres, as seen from the Earth, is the result of not just one, but several simultaneous processes, including Ceres' actual orbital motion, but also the orbital motion and daily rotation of the Earth. In addition, Gauss had to "correct" the observations, by taking account of the precession of the equinoxes (the slow shift of the Earth's rotational axis), optical aberration and refraction, etc.

Secondly, there is nothing in the observations of Ceres *per se,* which gives us any direct hint, about how distant the object might be from the Earth. Each observation defines nothing more than a "line-of-sight," a direction in which the object was seen. We can represent this situation as follows **(Figure 11.1)**: From each of three points, E_1, E_2, E_3, representing the positions of the Earth (or more precisely, of Piazzi's observatory) at the three times of observation, draw "infinite" lines L_1, L_2, L_3, each in

Figure 11.1. *Points P_1,P_2,P_3 must lie on lines of sight L_1,L_2,L_3. But where?*



Box I. The position of P_2 is related to that of P_1 and P_3, by a parallelogram, formed from displacements OQ_1 and OQ_3, along the axes OP_1 and OP_3, respectively.

Points Q_1 and Q_3 divide the segments OP_1 and OP_3 according to proportions which can be expressed in terms of the triangular areas T_{12}, T_{23}, and T_{13}. In fact, from the discussion in Chapter 10, we know that

$$\frac{OQ_1}{OP_1} = \frac{T_{23}}{T_{13}}, \text{ and}$$

$$\frac{OQ_3}{OP_3} = \frac{T_{12}}{T_{13}}.$$

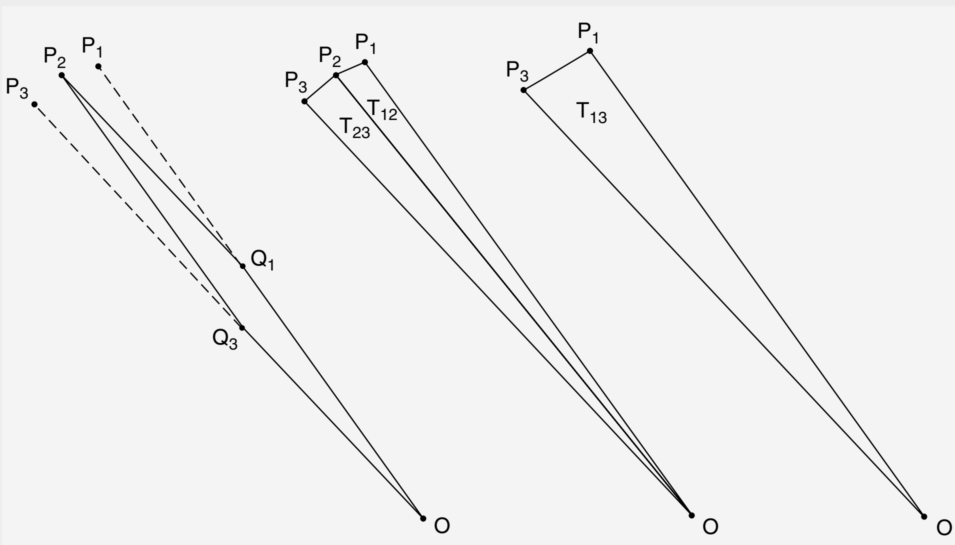
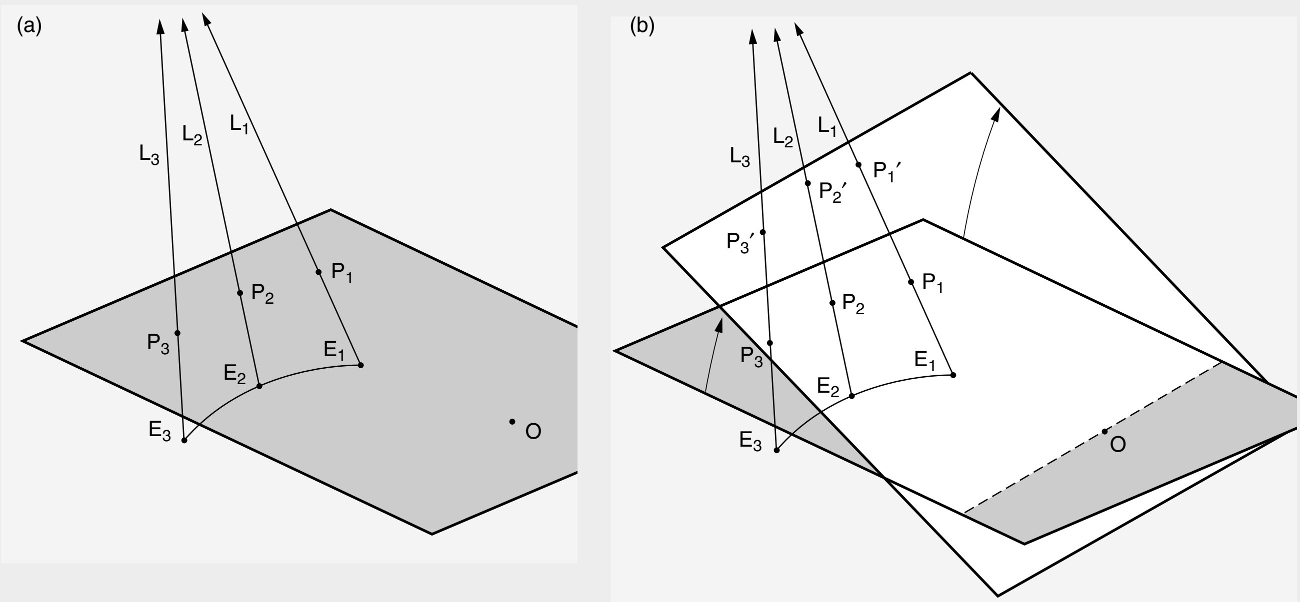


FIGURE 11.2. *(a) P_1, P_2, P_3, which are positions on Ceres' orbit, must all lie in some plane passing through the sun. (b) Each hypothetical position of the orbital plane defines a different configuration of positions P_1, P_2, P_3 relative to each other.*



the direction in which Ceres was seen at the corresponding time. Concerning the actual positions in space of Ceres (positions we have designated P_1, P_2, P_3), the observations tell us only, that P_1 is located somewhere along L_1, P_2 is somewhere on L_2, and P_3 is somewhere on L_3. For an empiricist, the distances along those lines remain completely indeterminate.

We, however, know more. If Ceres belongs to the solar system, its motion must be governed by the harmonic ordering of that system, as expressed (in part) by the Gauss-Kepler constraints. Those constraints reflect the curvature of the space-time, within which the events recorded by Piazzi occurred, and relative to which we must "read" his observations.

According to Gauss's first constraint, the orbit of Ceres is confined to some *plane* passing through the center of the sun. This simple proposition, should already transform our "reading" of the observations. The three positions P_1, P_2, P_3, rather than simply lying "somewhere" along the respective lines, are the points of intersections of the three lines L_1, L_2, L_3 with a certain plane passing through the sun. **(Figure 11.2a)** We don't yet know which plane this is; but, the very occurrence of an intersection of that form, already greatly reduces the degree of indeterminacy of the problem, and introduces a relationship between the three (as yet unknown) positions and distances.

Indeed, imagine a variable plane, which can pivot around the center of the sun; for each position of that plane, we have three points of intersection, with the lines L_1, L_2, L_3. Consider, how the *configuration* of those three points, relative to each other and the sun, *changes* as a function of the variable "tilt" of the plane. **(Figure 11.2b)** Can we specify something characteristic about the geometrical relationship among the three actual positions P_1, P_2, P_3 of Ceres, which might distinguish that specific group of points *a priori* from all other "triples" of points, generated as intersections of the three given lines with an arbitrary plane through the center of the sun?

Thanks to the work of the last chapter, we already have part of the answer. **(Box I)** We found, that the second position of Ceres, P_2, is related to the first and third positions P_1 and P_3, by the existence of a *parallelogram,* whose vertices are O, P_2, and two points Q_1 and Q_3, lying on the axes OP_1 and OP_3 respectively. Furthermore, we discovered that the *positions* Q_1 and Q_3, defining those two displacements, can be precisely characterized in terms of ratios of the triangular areas spanned by the positions P_1, P_2, P_3 (and O).

Henceforth, we shall sometimes refer to the values of those ratios, $T_{23}:T_{13}$ and $T_{12}:T_{13}$ (or, T_{23}/T_{13} and T_{12}/T_{13}), as "coefficients," determining the interrelation of the three positions in question.

We already observed in the last chapter, that the triangular areas entering into these relationships, are *nearly* the same as the orbital sectors swept out by the planet in moving between the corresponding positions; and, whose ratios are *known* to us, thanks to Kepler's "area law," as ratios of elapsed times. In fact, we calculated them in the last chapter from Piazzi's data.

The area of each orbital sector, however, *exceeds* that of the corresponding triangle, by the lune-shaped area, enclosed between the orbital arc and the straight-line segment connecting the corresponding two positions of the planet.

As long as the three positions of the planet are relatively close together—as they are in the case of Ceres at the times of Piazzi's observations—the lune-shaped excesses amount to only a small fraction of the areas of the triangles (or sectors). In that case, the ratios of the triangles $T_{23}:T_{13}$ and $T_{12}:T_{13}$ would be "very nearly" equal to the ratios of the corresponding orbital sectors, $S_{23}:S_{13}$ and $S_{12}:S_{13}$, whose values we calculated in the preceding chapter.

Can we regard the small difference between the triangle and sector ratios, as an "acceptable margin of error" for the purposes of a first approximation? If so, then we could take the numerical values calculated in Chapter 10 from the ratios of the elapsed times, and say:

$$\frac{T_{23}}{T_{13}} = \text{(approximately)} \; \frac{S_{23}}{S_{13}} = 0.513\,,$$

$$\frac{T_{12}}{T_{23}} = \text{(approximately)} \; \frac{S_{12}}{S_{23}} = 0.487\,.$$

Let us suppose, for the moment, that these equations were exactly correct, or very nearly so. What would they tell us, about the configuration of the three points P_1, P_2, P_3?

To get a sense of this, readers should perform the following graphical experiment: Choose a fixed point O, to represent the center of the sun, and choose *any* two other points as hypothetical positions for P_1 and P_3. Next, determine the corresponding positions of Q_1 and Q_3 on the segments OP_1 and OP_3, so that OQ_1 is 0.513 times the total length of OP_1, and OQ_3 is 0.487 times the total length of OP_3. Combine the displacements OQ_1 and OQ_3 according to the "parallelogram law," to determine a position for P_2. Now, change the positions of P_1 and P_3, and see how P_2 changes. What remains constant in the relationship between P_2, P_1, and P_3? Also, examine the effect, of replacing the "coefficients" just used, by some other pair of values, say 0.6 and 0.9.

Evidently, by *specifying* the values of the ratios in terms of which the position of P_2 is determined by those of P_1 and P_3, we have *greatly restricted* the range of "possible" triples of points, which could qualify as the three actual positions for Ceres.

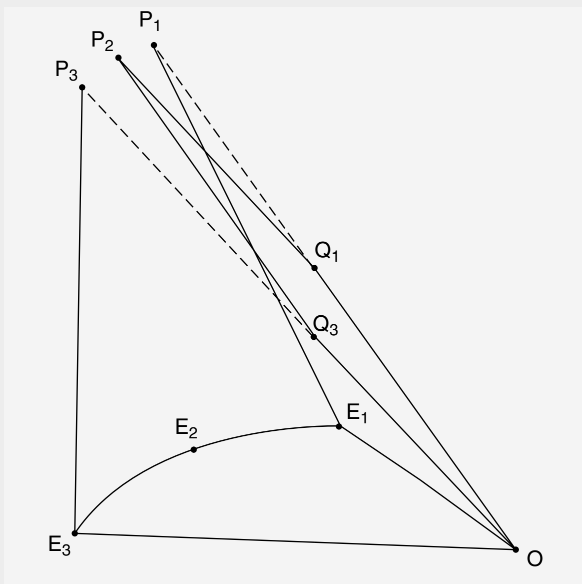
Recall the image of a manifold of "triples" of points, generated as the intersections of a variable plane, passing through the center of the sun, with the three "lines of sight" L_1, L_2, L_3. (SEE Figure 11.2) How many of those triples manifest the specific type of relationship of the second upon the first and third, defined by those specific values for the coefficients? Exploring this question by drawings and examples, we soon gain the conviction, that—apart from very exceptional cases in terms of the lines L_1, L_2, L_3, and the specified values of the coefficients—the specified type of configuration is realized for only *one* position of the movable plane. Thus, the positions of the three points in question, are practically uniquely determined, once L_1, L_2, L_3 and the "coefficients" are given.

If that is the case, then the task we have set ourselves must, intrinsically, be capable of solution! In particular, there must be a way to determine the Earth-Ceres distances from nothing more than the directions of the lines L_1, L_2, L_3 (as given by Piazzi's observations), the positions of the Earth, and sufficiently accurate values for the coefficients defined above.

To see how this might be accomplished, reflect on the implications of the parallelogram expressing the interrelationship between the second, and the first and third positions of Ceres. (SEE Box I) That parallelogram expresses the circumstance, that the (as yet unknown) position of P_2, results from a combination of the two displacements OQ_1 and OQ_3. Concerning the positions of Q_1 and Q_3, we know that they lie on the segments OP_1 and OP_3, respectively, and divide those segments according to proportions ("coefficients") whose values are known to us, at least in approximation. **(Figure 11.3)** Unfortunately, since we don't know P_1 and P_3, we have no way to directly determine the positions of Q_1 and Q_3 in space.

Let us look into the situation more carefully. Consider, first, the displacement OQ_1 in relation to the positions of the sun, Earth, and Ceres at the first moment of observation. Those positions form a triangle, whose sides are

FIGURE 11.3. *Closing in on P_2. The proportional relationships of Q_1, Q_3 to OP_1, OP_3 are known approximately from the ratios of elapsed times.*



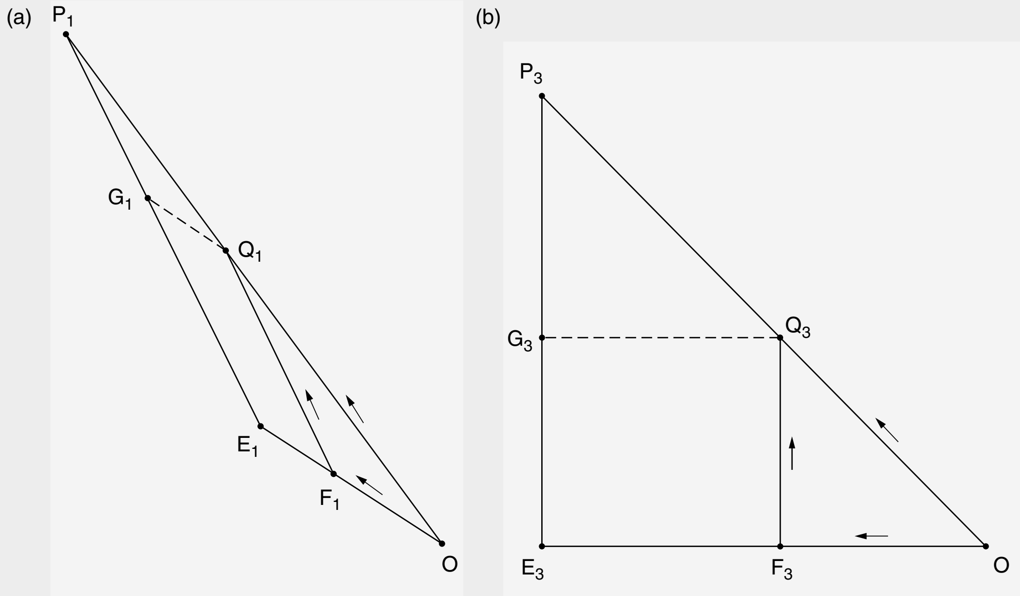


FIGURE 11.4. *Closing in on P_2. Determining (a) F_1 and the direction of F_1Q_1, and (b) F_3 and the direction of F_3Q_3.*

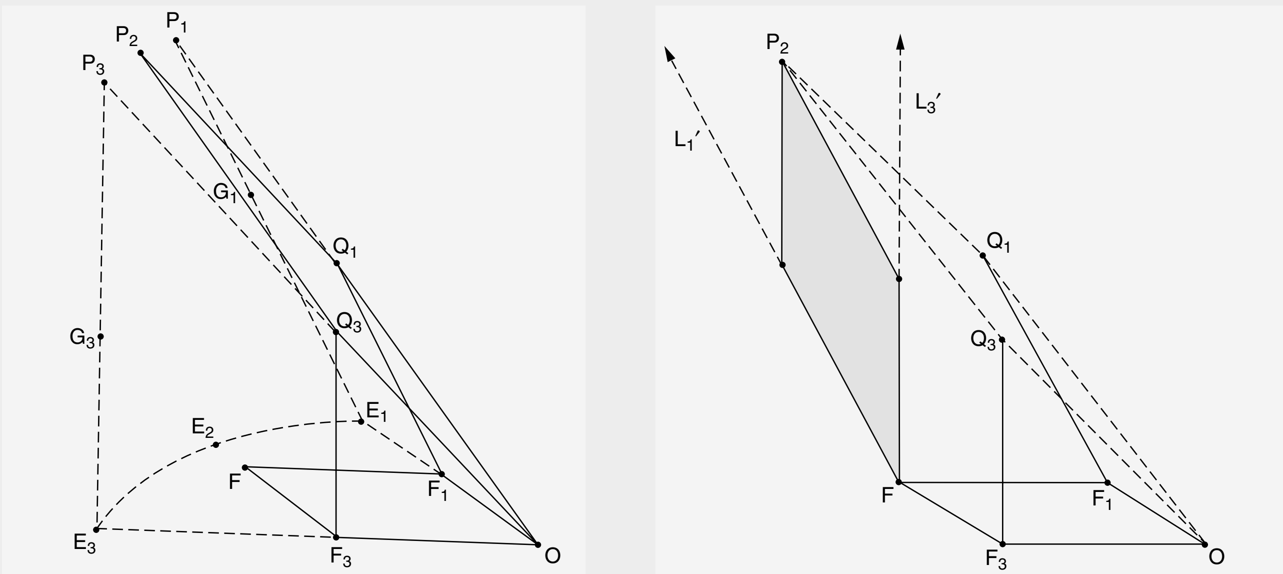
OE_1, OP_1 and E_1P_1. **(Figure 11.4a)** Point Q_1 lies on one of those sides, namely OP_1, dividing it according to the proportion defined by the first coefficient. However, we can't say anything about the lengths of OP_1 and E_1P_1, nor about the angle between them, so the position of Q_1 remains undetermined for the moment.

But what about the points, which correspond to Q_1 on the *other* sides of the triangle? Draw the parallel to the line-of-sight E_1P_1, through Q_1 down to OE_1. That parallel intersects the axis OE_1 at a location, which we shall call F_1. That point F_1 will divide the segment OE_1 by the *same* proportion, that Q_1 divides OP_1 (for, by construction,

BOX II. The position of P_2 results from the combination of the displacements OQ_1 and OQ_3. On the other hand, by our constructions,

$$OQ_1 = OF_1 + F_1Q_1, \text{ and } OQ_3 = OF_3 + F_3Q_3.$$

Combine displacements OF_1 and OF_3, to get a position F, and then perform the other two displacements, F_1Q_1 and F_3Q_3. This amounts to constructing a parallelogram based at F whose sides are parallel to, and congruent with, the segments F_1Q_1 and F_3Q_3. The directions of the latter segments are parallel to Piazzi's "lines of sight" from E_1 to P_1 and E_3 to P_3, respectively. The end result must be P_2. This tells us that P_2 lies in the plane through F, determined by those two "line of sight" directions.



OF_1Q_1 and OE_1P_1 are similar triangles). That proportion, as we noted, is at least approximately known. Since the position of the Earth, E_1, is known, *we can determine the position of* F_1 *directly,* by dividing the known segment OE_1 according to that same proportion.

This result brings us, by implication, a dimension closer to our goal! Observe, that—by construction—the segment F_1Q_1 is parallel to, and congruent with, a sub-segment of the line-of-sight E_1P_1. Call that sub-segment E_1G_1. In other words, to arrive at the location of Q_1 from O, we can first go from O to the position F_1, just constructed, and then carry out a second displacement, equivalent to the displacement E_1G_1 but applied to F_1 instead of E_1. We don't know the magnitude of that displacement, but we *do know its direction,* which is that of the line of sight L_1 given by Piazzi's first observation.

Now, apply the very same considerations, to the positions for the third moment of observation (i.e., the triangle OE_3P_3). **(Figure 11.4b)** Dividing the segment OE_3 according to the value of the second coefficient, determine the position of a point F_3 on the line OE_3, such that the line F_3Q_3 is parallel with the line-of-sight E_3P_3. The displacement OQ_3 is thus equivalent to the combination of OF_3, and a displacement in the direction defined by the line of sight E_3P_3, i.e., L_3.

We are now inches away from being able to determine the position of P_2! Recall, that we resolved the displacement OP_2 into the combination of OQ_1 and OQ_3. Each of the latter two displacements, on the other hand, has now been decomposed, into a known displacement (OF_1 and OF_3, respectively), and a displacement along one of the directions determined by Piazzi's observations. In other words, OP_2 is the result of *four* displacements, of which two are known in direction and length, and the other two are known only as to direction. **(Box II)**

Assuming, as we did from the outset, that *the result of a series of displacements of this type, does not sensibly depend on the order in which they are combined,* we can imagine carrying out the four displacements, yielding the position of P_2 relative to O, in the following way: First, combine the displacements OF_1 and OF_3. The result is a point F, located in the plane of the ecliptic. We can determine the position of F directly from the known positions F_1 and F_3. Then, apply the two remaining displacements, to get from F to P_2.

What does that say, about the nature of the relationship of P_2 to F? We don't know the magnitudes of the displacements carrying us from F to P_2, but we know their two directions. They are the directions defined by Piazzi's original lines of sight, L_1 and L_3. *Aha!* Those two directions, as projected from F, define a *specific plane* through F. We have only to draw parallels L_1', L_3' through F, to the just-mentioned lines of sight; the plane in question, plane Q, is the plane upon which L_1' and L_3' lie. **(Figure 11.5)** Since that plane contains both of the directions of the two displacements in question, their combined result, starting from F—i.e., P_2—will in any

FIGURE 11.5. *P_2 must lie on plane Q constructed at point F. But where?*

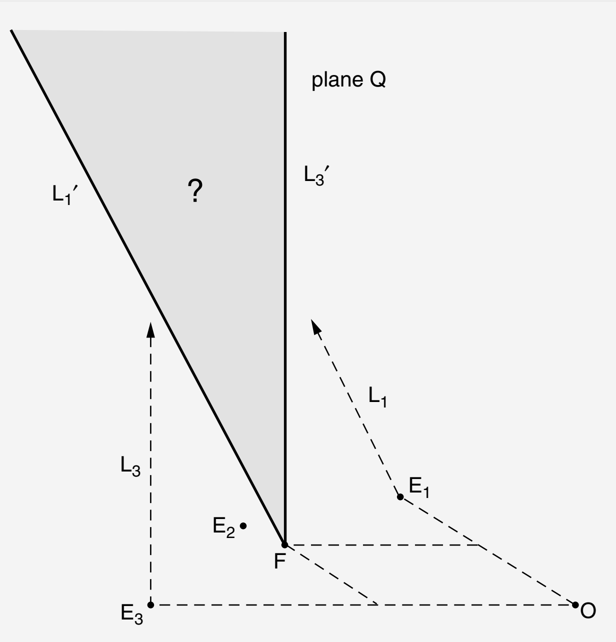
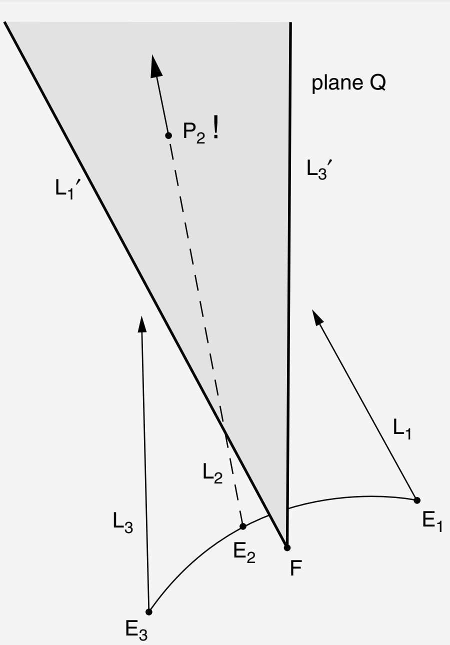


FIGURE 11.6. *Locating P_2. Line L_2, originating at E_2, must intersect plane Q at point P_2. E_2P_2 is the crucial distance we are seeking.*



case be some point in that plane.

So, P_2 lies on that plane. But where? Don't forget the second of the selected observations of Piazzi! That observation defines a line L_2, extended from E_2, along which P_2 is located. Where is it located? Evidently, *at the point of intersection of L_2 with the plane which we just constructed!* **(Figure 11.6)** The distance along L_2, between E_2 and that point of intersection (i.e., the distance E_2P_2), is the crucial distance we are seeking. *Eureka!*

This—with one, *very crucial* addition by Gauss—defines the kernel of a method, by which we can actually calculate the Earth-Ceres distance. It is only necessary to translate the geometrical construct, just sketched, into a form which is amenable to precise computations.

However, the pathway of solution we have found so far, has one remaining flaw. We shall discover that, and Gauss's ingenious remedy, in Chapter 12.

In the meantime, readers should ponder the following: The possibility of determining the position of P_2, as the intersection of the line L_2 with a certain plane through F, presupposes, that F does not coincide with the origin of that line, namely E_2. In fact, the size of the gap between F and E_2, reflects the difference in curvature between the orbits of Earth and Ceres, over the interval from the first to the third observations.

—*JT*

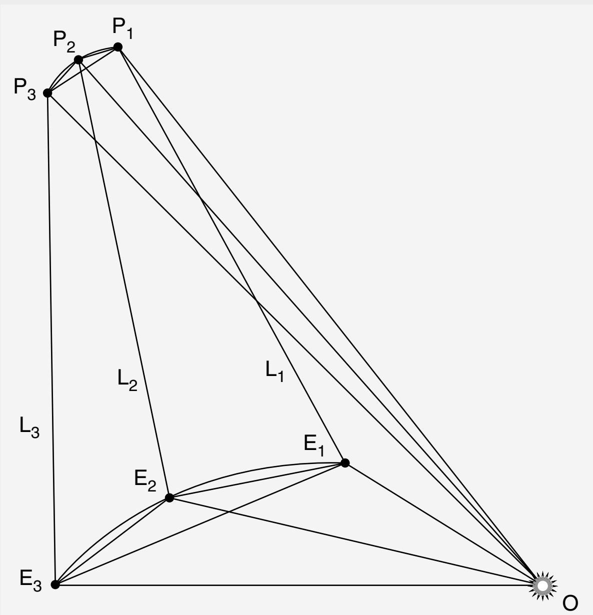
CHAPTER 12

An Unexpected Difficulty Leads to New Discoveries

In Chapter 11, we appeared to have won a major battle in our efforts to determine the orbit of Ceres from three observations. The war, however, has not yet been won. As we soon shall see, the greatest challenge still lies before us.

We developed a geometrical construction that gives us an approximation for the second position of Ceres. That construction consisted of the following essential steps:

FIGURE 12.1. *Relationships of the positions of the sun (O), Earth (E_1,E_2,E_3), lines of sight, and Ceres (P_1,P_2,P_3).*



1. The three chosen observations define the directions of three "lines-of-sight" from Piazzi's observatory through the positions of Ceres, at each of the given times of observation. Using that information, and the known orbit and rotational motion of the Earth, determine the positions of the observer, E_1, E_2, E_3, and construct lines L_1, L_2, L_3, running from each of those positions in the corresponding directions.* **(Figure 12.1)**

2. From the times provided for Piazzi's observations, compute the ratios of the elapsed times, between the first and second, the second and third, and the first and third times—i.e., the ratios $t_2 – t_1 : t_3 – t_1$ and $t_3 – t_2 : t_3 – t_1$.

3. According to Kepler's "area law," the values, just computed, coincide with the ratios of the sectoral areas, $S_{12} : S_{13}$ and $S_{23} : S_{13}$, swept out by Ceres over the corresponding time intervals. We *assumed,* that for the pur-

* For reference, Piazzi gave the apparent positions for Jan. 2, Jan. 22, and Feb. 11, 1801, as follows:

	right ascension	declination
Jan. 2	51° 47′ 49″	15° 41′ 5″
Jan. 22	51° 42′ 21″	17° 3′ 18″
Feb. 11	54° 10′ 23″	18° 47′ 59″

Those "positions" are nothing but the directions in which the lines L_1, L_2, L_3 are "pointing."

pose of approximation, it would be possible to ignore the relatively small discrepancy between the ratios of the *orbital sectors* on the one hand, and those of the corresponding *triangular areas* formed by the sun and the corresponding positions of Ceres, on the other. **(Figure 12.2)**

4. On that basis, we assumed that the ratios of the elapsed times, computed in step **2**, provide "sufficiently precise"*approximations* to the values for the ratios of the triangular areas, $T_{12}:T_{13}$ and $T_{23}:T_{13}$. The true values of those ratios, which I shall refer to as "*d*" and "*c*," respectively, are the coefficients which define the spatial relationship of the *second* position of Ceres to the *first* and *third* positions, in terms of the "parallelogram law" for the combination and decomposition of simple displacements in space.

5. Using the approximate values for *c* and *d* adduced from the elapsed times in the manner just described, construct a position *F*, in the plane of the Earth's orbit, in such a way, that *F's relationship to the Earth positions* E_1 *and* E_3*, is the same as that adduced to exist between the second, and first and third positions of Ceres.*

To spell this out just once more: Divide the lengths of the segments from the sun to the Earth, OE_1 and

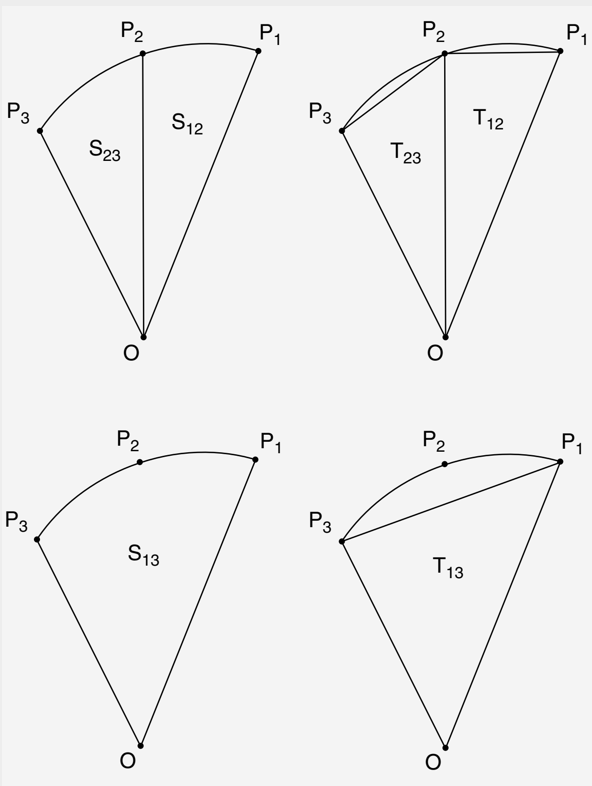


FIGURE 12.2. *Orbital sectors* S_{12}, S_{23}, S_{13} *and corresponding triangular areas* T_{12}, T_{23}, T_{13}.

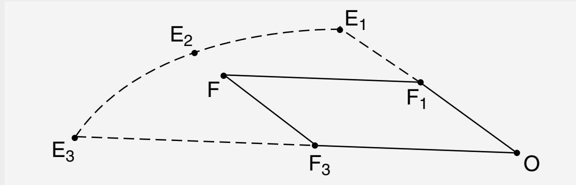


FIGURE 12.3. *Determining point F, as a combination of displacements along* OE_1 *and* OE_3.

OE_3, according to the ratios defined by the approximate values for the coefficients *c* and *d*. In other words, construct points F_1 and F_3, along the segments OE_1 and OE_3, respectively, in such a way, that $OF_1/OE_1 = t_3 - t_2 / t_3 - t_1$ and $OF_3/OE_3 = t_2 - t_1 / t_3 - t_1$. Then, construct *F* as the endpoint of the resultant of the two displacements OF_1 and OF_3. (Thus, *F* will be the fourth vertex of the parallelogram constructed from points *O*, F_1, and F_3.) **(Figure 12.3)**

6. Next, draw lines L_1', L_3' parallel to the lines L_1 and L_3, through *F*. The resulting lines determine a unique plane, *Q*, passing through *F*.

7. Determine the point *P*, where the line L_2 intersects the plane *Q*. In other words, "project" from the second position E_2 of the Earth, along the "line of sight" defined by the second observation, until you hit the plane *Q*. **(Figure 12.4)** That point, *P*, is our first approximation for the Ceres position P_2!

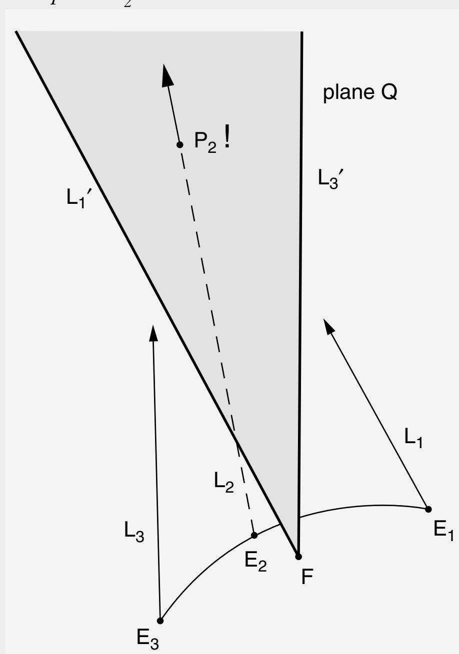


FIGURE 12.4. *The intersection of line* L_2 *with plane Q, determines point* P_2.

Using routine methods of analytical and descriptive geometry, as developed by Fermat and perfected by Gaspard Monge *et al.,* we can translate the geometrical construction, sketched above, into a procedure for numerical computation of the distance E_2P, from the data provided by Piazzi.

We would be well advised, however, to think twice before launching into laborious calculation. As it stands, our method is based on a crude approximation for estimating the values of the crucial coefficients, *c* and *d*. Remember, we chose to ignore the differences between the orbital sectors and the corresponding triangles. We might argue for the admissibility of that step, for the purposes of approximation, as follows.

Firstly, we are concerned only with the ratios, and not the absolute magnitudes of the sectors and triangles. Secondly, the differences in question—namely the lune-shaped areas contained between the orbital arcs and the straight-line chords connecting the corresponding orbital positions—are certainly only a tiny fraction of the *total* areas of the orbital sectors. Hence, they will have only a "marginal" effect on the values of the *ratios* of those areas.

In fact, simple calculations, carried out for the hypothetical assumption of a circular orbit between Mars and Jupiter,* indicate, that we can expect an error on the order of about *one-fourth of one percent* in the determination of the coefficients *c* and *d*, when we disregard the difference between the sectors and the triangles. Not bad, eh?

Before celebrating victory, however, let us look at the possible effect of that magnitude of error in the coefficients, for the rest of the construction.

Look at the problem more closely. An error of *x* percent in the values of *c* and *d*, will produce a corresponding percentual error in the positions of F_1 and F_2, and at most twice that error, in the process of combining OF_1 and OF_3 to create *F*. Any error in the position of *F*, however, produces a corresponding shift in the position of the plane *Q*, whose intersection with L_2 defines our approximation to the position of Ceres. Now, the *directions* of the lines L_1, L_2, L_3, which arose from observations made over a relatively short time, differ only by a few degrees. Since the orientation of the plane *Q* is determined by parallels to L_1 and L_3 at *F*, this means that L_2 will make an extremely "flat" angle to the plane *Q*. A slight shift in the position of the plane, yields a *much larger* change in the location of its intersection with L_2. How much larger? If we analyze the relative configuration of L_2, *Q*, and the ecliptic, corresponding to the situation in Piazzi's observations, then it turns out that any error in the position of *F*, can generate an error *ten to twenty times larger* in the location of the intersection-point. **(Figure 12.5)** That would bring us into the range of a worrisome 5-10 percent error in our estimate for the Earth-Ceres distance E_2P_2.

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* To get a sense, how large that supposedly "marginal" error might be, let us work out a hypothetical case. Suppose that the unknown planet were moving in a circular orbit, about halfway between Mars and Jupiter; say, at a distance of 3 Astronomical Units (A.U.) from the sun (three times the mean Earth-sun distance). According to Kepler's constraints, the square of the periodic time (in years) of any closed orbit in the solar system, is equal to the cube of the major axis of the orbit (in A.U.). The periodic time for the unknown planet, in this case, would be the square root of 3×3×3, or about 5.196152 (years). In a period of 20 days (i.e., approximately the time between the successive observations selected by Gauss), the planet would traverse a certain fraction of a total revolution around the sun, equivalent to 20 divided by the number of days in the orbital period of 5.196152 years, i.e., 20/(365.256364 × 5.196152), or 0.010538. To find the area of the orbital sector swept out during 19 days, we have only to form the product of 0.010538 and the area enclosed by a total revolution—the latter being equal to π (~3.141593) times the square of the orbital radius (3×3). We get a result of 0.297951, in units of square A.U.

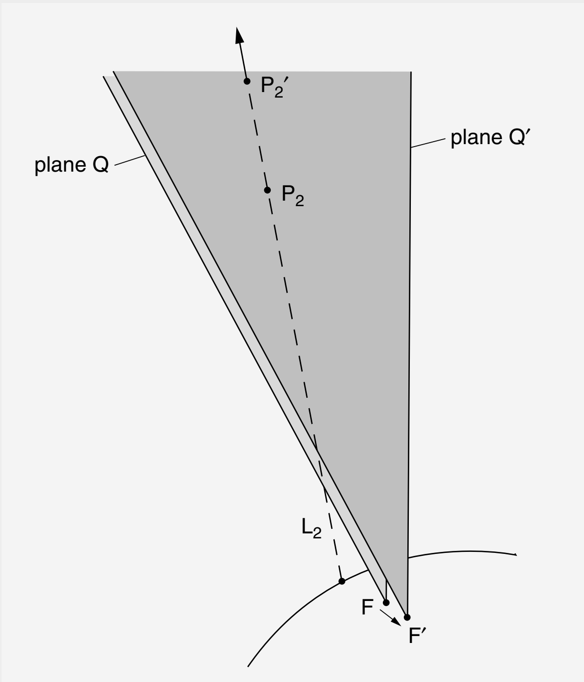
Next, compute the triangular area between the sun and two positions of the planet, 20 days apart. The angle swept out at the sun by that motion, is 0.010538 × 360°, or 3.79368°. The height and base of the corresponding isosceles triangle, whose longer sides are equal to the orbital radius, can be estimated by graphical means, or computed with the help of sines and cosines. The triangle is found to have a height of 2.998356 A.U. and a base (the chord between the two planetary positions) of 0.198600 A.U., for an area of 0.297737 square A.U.

Comparing the values just obtained, we find the excess area of the orbital sector over the triangle, to be a "mere" 0.000214 square A.U. (Given that an astronomical unit is 150 million kilometers, that "tiny" area corresponds to "only" about 5 trillion square kilometers!) More to the point, the ratio of the sector to the triangular area is 1.000718. Thus, in replacing the triangular areas T_{12} and T_{23} by the corresponding sector areas S_{12} and S_{23}, in the ratios which define the coefficients *c* and *d*, we introduce an error of about 0.07 percent.

Note, however, that these estimates only apply to an elapsed time of the order of 20 days—such as between the first and second, and the second and third positions. The first and third positions, on the other hand, are about 40 days apart; calculating this case through, we find an orbital sector area of 0.595902 and a triangular area of 0.594170 square A.U. In this case, the difference is 0.00193 square A.U.—almost *eight times* what it was in the earlier case!—and the ratio is 1.0029, corresponding to a proportional error of more than 0.29 percent. This is the error to be expected, when we use S_{13} instead of T_{13} in the ratios defining the coefficients *c* and *d*.

From these exploratory computations, we conclude that by far the largest source of error, in our estimate of the coefficients *c* and *d*, is due to the discrepancy between S_{13} and T_{13}.

FIGURE 12.5. *Owing to the extremely flat angle which the line L_2 makes to the plane Q, a slight shift in the position of F (from F to F′) causes a much larger change in the point of intersection with L_2 (from P_2 to P_2').*



As a matter of fact, our calculation with circular orbits *greatly* underestimates the error in the coefficients *c* and *d*, which would occur in the case of a significantly *non-circular* orbit (as is the case for Ceres). In that case, the error can amount to 2 percent or more, leading to a final error of 20-30 percent in our estimate of the object's distance.

Such a huge margin of error would render any prediction of the position of Ceres completely useless.

Back to Curvature

Reality has rejected the crudeness of our approach, in trying to ignore the discrepancies between the orbital sectors and the corresponding triangles. Those discrepancies are, in fact, the most crucial characteristics of the orbit itself "in the small"; they result from the curvature of the orbit, as reflected in the elementary fact, that the path of the planet between any two points, no matter how close together, is always "curving away from" a straight line.*

To come to grips with the problem, no less than *three levels* of the process must be taken into account:

(i) The curvature "in the infinitely small," which acts in any arbitrarily small interval, and continuously "shapes" the orbit at every moment of an ongoing process of generation.

(ii) The curvature of the orbit "in the large," considered as a "completed" totality "in the future," and which ironically pre-exists the orbital motion itself; this, of course as defined in the context of the solar system as a whole.

(iii) The geometrical intervals among discrete loci P_1, P_2, etc., of the orbit, as moments or events in the process, and whose relationships embody a kind of *tension* between the apparent cumulative or integrated effect of curvature "in the small," and the curvature "in the large"—acting, as it were, from the future.

Euler, Newton, and Laplace rejected this, linearizing both in the small and in the large. From the standpoint of Newton and Laplace, the orbit *as a whole*—history!—has no *efficient* existence. An orbit is only the accidental trace of a process which proceeds "blindly" from moment to moment under the impulse of momentary "forces"—like the "crisis management" policies of recent years! For the Newtonian, only "force", which you can "feel" in the "here and now," has the quality of reality. But Newtonian "blind force" is a purely linear construct, devoid of cognitive content. You can travel the entropic pathway of deriving the "force law" algebraically from Kepler's Laws; but, in spite of elaborate efforts of Laplace *et al.,* it is axiomatically impossible to derive the Keplerian ordering of the solar system as a whole, from Newton's physics.

In fact, the efforts of Burkhardt and others, to determine the orbit of Ceres using the elaborate mathematical apparatus set forth by Laplace in his famous *Méchanique Céleste,* proved a total failure. According to the report of Gauss's friend, von Zach, the elderly Laplace, who—from the lofty heights of Olympus, as it were—had been following the discussions and debates concerning Ceres, concluded that it was *impossible* to determine the orbit

* Industrious readers, who took the trouble to actually plot the position of *F*, using the ratios of elapsed times as described above, will have discovered, that *F* lies on the *straight line* between E_1 and E_3. One might also note the following:

(i) As long as we use the ratios of elapsed times as our coefficients, the sum of those coefficients will invariably be equal to 1.

(ii) If we have any two points *A* and *B*, divide the segments *OA* and *OB* according to coefficients whose sum is equal to 1, and generate the corresponding displacements along those two axes. The point resulting from the combination of those displacements, will always lie along the straight line joining *A* and *B*.

(iii) Consequently, insofar as P_2 does *not* lie on the segment P_1P_3, in virtue of the curvature of Ceres' orbit, the sum of the *true* values of *c* and *d*, will always be different from, and, in fact, greater than 1.

from Piazzi's limited data. Laplace recommended calling off the whole effort, waiting until some astronomer, by luck, might succeed in finding the planet again. When von Zach reported the results of Gauss's orbital calculation, and the extraordinary agreement between Gauss's proposed orbit and the entire array of Piazzi's observations, this was pooh-poohed by Laplace and his friends. But reality soon proved Gauss right.

Characteristic of the axiomatic superiority of Gauss's method, as of Kepler before him, is that Gauss treats the orbits as efficient entities. Accordingly, let us investigate the relationships among P_1, P_2, P_3, which necessarily ensue from the fact that they are subsumed as moments of a unique Keplerian orbit.

A Geometric Metaphor

For this purpose, construct the following representation of the manifold of all potential orbits (seen as "completed" totalities), having a common focus at the center of the sun, and lying in any given plane. **(Figure 12.6)** Represent that plane as a horizontal plane, passing through a point *O*, representing the center of the sun. Above the plane, generate a circular cone, whose vertex is at *O*, and whose axis is the perpendicular to the plane through *O*.

Cutting the cone by another, variable plane, we generate the entire array of conic sections. The perpendicular projection of each such conic section, down onto the horizontal plane, will also be a conic section; and the resulting conic sections in the horizontal plane will all have the point *O* as a common focus.* (SEE "The Ellipse as a Conical Projection," in the **Appendix**)

This construction can be "read" as a geometrical metaphor, juxtaposing two different "spaces" that are axiomatically incompatible. In this metaphor, the cone represents the invisible space of the process of creation (which Lyndon LaRouche sometimes calls the "continuous manifold"), while the horizontal plane represents the space of visible phenomena. The projected conic section is the visible, "projected" image of a singularity in the higher space.

Using this construction, examine the relationship among P_1, P_2, P_3, and the unique orbit upon which P_1, P_2, P_3 lie. We can determine that orbit by "inverse projection," as follows. **(Figure 12.7)**

At each of P_1, P_2, P_3, draw a perpendicular to the horizontal plane. Those three perpendiculars intersect the cone at corresponding points, U_1, U_2, U_3. The latter points, in turn, determine a unique plane, cutting the cone through those points and generating a conic section containing them. The projection of that conic section onto the "visible" horizontal plane, will be the unique orbit upon which P_1, P_2, P_3 lie. Note, that the heights h_1, h_2, h_3 of the points U_1, U_2, U_3 above the horizontal plane are proportional to the radial distances of P_1, P_2, P_3 from the origin *O*.

Note an additional singularity, generated in the process: The plane through U_1, U_2, U_3 intersects the axis of the cone at a certain point, *V*. The *height* of that point on the axis above *O*, is, in fact, closely related to the

* I first presented the basic idea of this construction in an unpublished April 1983 paper entitled "Development of Conical Functions as a Language for Relativistic Physics."

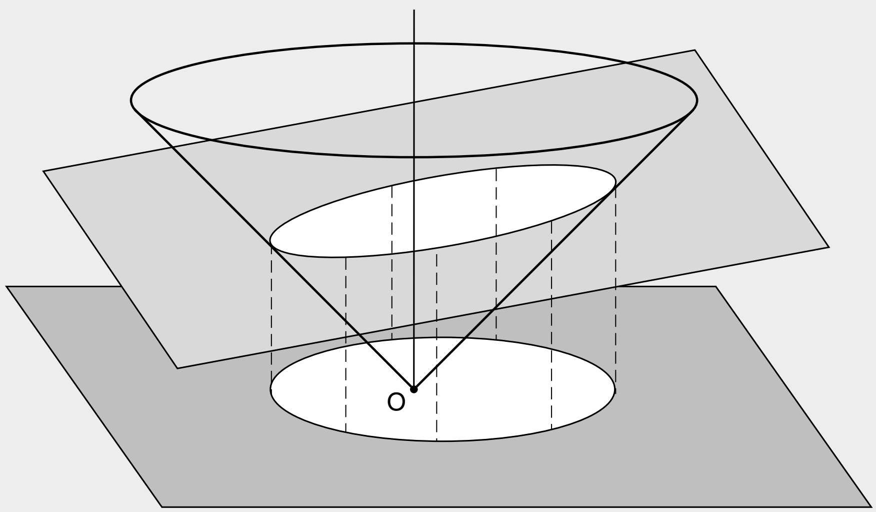


FIGURE 12.6. *Construct a circular cone with apex at point O, the position of the sun in a horizontal plane. By cutting the cone with a second plane, we generate an ellipse. When projected down onto the horizontal plane, this ellipse will generate a corresponding, second ellipse. We shall use this construction to investigate the relationships of the orbital sectors and triangular areas formed by the observed positions of Ceres.*

FIGURE 12.7. *Use our construction to relate positions P_1,P_2,P_3, radial distances r_1,r_2,r_3, heights h and h_1,h_2,h_3, and the orbital parameter.*

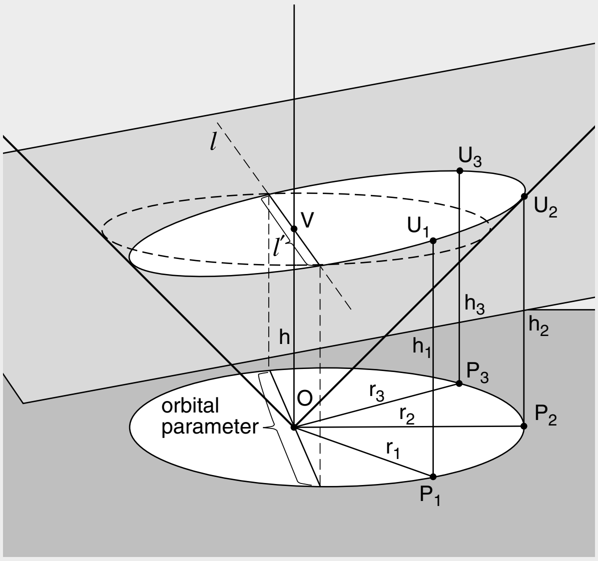
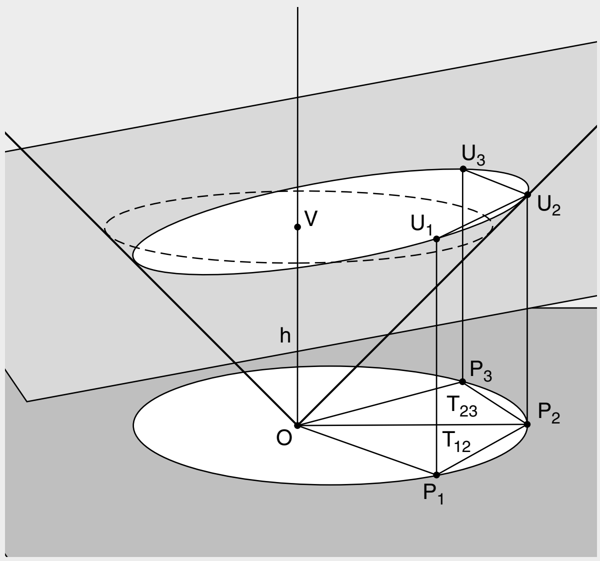


FIGURE 12.8. *What is the relationship between the triangular areas T_{12}, T_{23}, T_{13} and height h of the point V?*



"parameter" of the orbit, which played a key role in Gauss's formulation of Kepler's constraints. Gauss showed, that the area swept out by a planet in its motion in a given orbit over any interval of time, is proportional (by a universal constant of the solar system) to the duration of the time interval, multiplied by the square root of the "orbital parameter." Integrating this with the conical representation that we have just introduced, opens up a new pathway toward the solution of our problem.

In fact, if we cut the cone horizontally at the height of V, then the intersection of that horizontal with the plane of U_1, U_2, U_3, will be a line l, perpendicular to the main axis of the conic section. That line l intersects the conic section in two points, which lie symmetrically on opposite sides of V and at the same height. The segment l' of l (bounded by those points) defines the cross-width of the conic section at V. Line segment l' is also a diameter of the cone's circular cross-section at V, which in turn is proportional to the height h of V on the axis. Now, project down to the horizontal plane of P_1, P_2, P_3. The image of l', equivalent to l' in length, is the perpendicular diameter of the orbit at the focus O, exactly the length that Gauss called the "parameter" of the orbit.

All of this can be seen, nearly at a glance, from the diagram in Figure 12.6. The immediate upshot is, that Gauss's "orbital parameter," which governs the relationship between the elapsed time and the area swept out by the motion of a planet in its orbit, is proportional to the h of the point V on the axis of the cone.

On the other hand, our method of "inverse projection" allows us to determine V *directly* in terms of the three positions P_1, P_2, P_3, by constructing the plane through the corresponding points U_1, U_2, U_3. As a "spin-off" of these considerations, we obtain a simple way to determine Gauss's orbital parameter for any orbit, from nothing more than the positions of any three points on the orbit. We can say even more, however.

We found, earlier, a way to express the spatial relationship between P_1, P_2, P_3 (relative to O), in terms of the ratios of the triangular areas T_{12}, T_{23}, T_{13}. This points to the existence of a simple *functional relationship* between those triangular areas, and the value of the orbital parameter (or, equivalently, the height of V). The latter, in turn, is functionally related to the values of corresponding times and orbital sectors, by Gauss's constraint. **(Figure 12.8)**

Our conical construction has provided a missing link, in the necessary coherence of the orbital sectors with the corresponding triangles. This, in turn, will allow us to supersede the crude approximation, used so far, and to determine the Ceres distance with a precision which Laplace and his followers considered to be impossible.

The details will be worked out in the following chapter. But, it is already clear, that we have advanced by another, critical dimension, closer to victory. The key to our success, was a sortie into the "continuous manifold" underlying the planetary orbits.

—JT